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Combined Accumulation- and Decumulation-Plans with Risk-Controlled Capital Protection

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Abstract

We base our analysis on an investor, usually a retiree, endowed with a certain amount of wealth W, who considers both his own consumption needs (fixed periodic withdrawals) and the requirement of his heirs (defined bequest). For this purpose he pursues the following investment strategy. The part F is invested in a set of investment funds with the target to achieve an accumulated wealth at the end of a certain time horizon of at least the original amount of wealth W (or the fraction $(1-h)W$), measured in real terms. As certain investment risks are implied, we allow for the probability of falling short of the target and implement it into our model as a risk control parameter. The remaining part MM of the original wealth is invested in money market funds in order to avoid additional investment risks and deliver fixed periodic withdrawals until the end of the respective time horizon. The optimal investment strategy is the investment fund allocation that satisfies the probability of shortfall and minimizes F, while maximizing the fixed periodic withdrawals. We outline this investment problem in a mathematical model and illustrate the solution for a reasonable choice of empirical parameters.

Keywords:

1. Investment problem

An investor of a certain age, usually a retiree of about 60 years, possesses a certain amount of wealth W, e.g. 100.000€, which he invests according to the following requirements:

- A minimal part F is invested in a set of investment funds, or asset categories, with the target to achieve an accumulated wealth at the end of a certain time horizon, e.g. 20 years of at least the original amount of wealth W or the fraction $(1-h)W$, measured in real terms (capital protection in real terms for a defined bequest).
- The remaining part MM of the original wealth is invested in money market funds, out of which an annual annuity due is withdrawn until the end of the respective time horizon (annuitization for individual consumption needs).

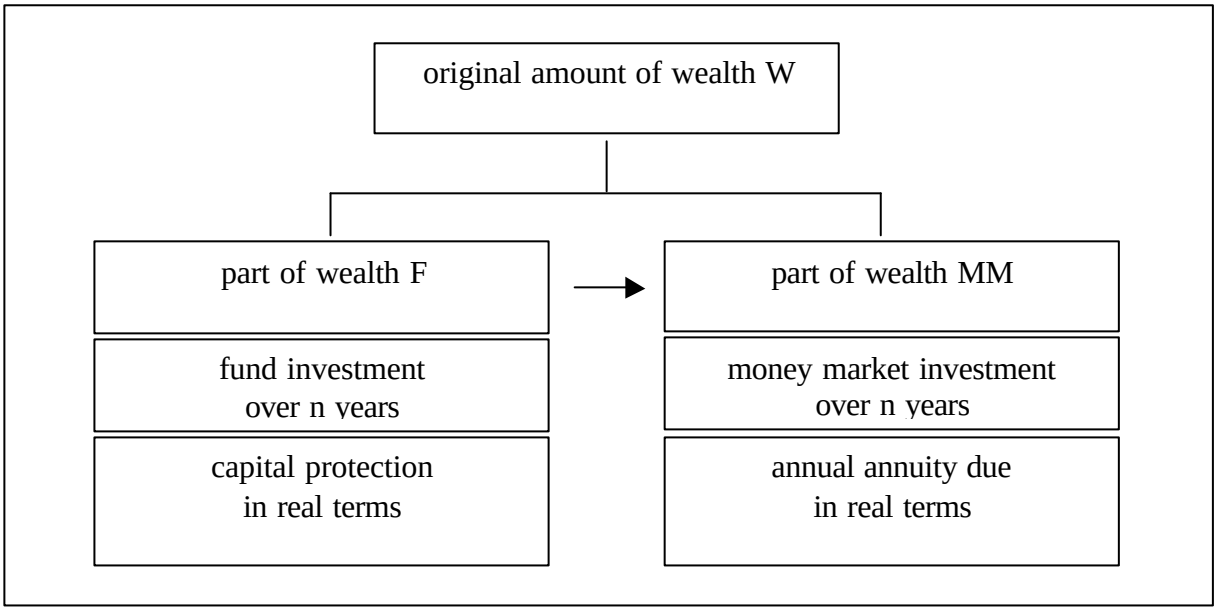

Figure 1: illustration of the investment problem

It is evident, that the amount of F determines the amount of MM and therefore also the annuity due. In order to maximise the annuity due, the investor has to choose an investment strategy, minimizing the amount of F, while meeting the above-mentioned investment requirements.

In the following, we develop a general solution for this respective investment problem. For clarity reasons, we present our model in detail in the appendix against a theoretical background.

2. Methodology

2.1 Condition of risk-controlled capital protection in real terms

In case of fund investment under risk, the reach of the respective investment target is not only determined by the average investment return, but also by the volatility of the fund. Therefore, it is necessary to specify a condition, that incorporates the capital protection for a defined bequest under risk. Capital protection for fund investment under risk can not be guaranteed with full certainty, but only to a distinct degree of certainty, being represented by a probability.

Thus, we propose the following criterion of risk control based on the shortfall probability. This condition of risk-controlled capital protection in real terms is orally defined as:

At the end of a previously fixed time horizon, the desired fraction of the original amount of wealth $(1-h)W$ *may fall short merely in a maximum of* α *out of 100 investment outcomes.*

The parameter α is a confidence coefficient, that has to be individually defined by the investor, e.g. $\alpha = 1\%, 5\%, 10\%$. This means, that the desired fraction of wealth is failed in no more than 1%, 5% or 10% of all possible investment scenarios. In this way, the *shortfall probability* of the desired fraction of capital protection can be controlled. The determination of the Value-at-Risk of the distribution of wealth at the end of the time horizon constitutes the focus of our methodology. For a mathematical formalization, the reader is referred to the appendix.

Figure 2 summarizes the general procedure of our formalization in order to generate the minimal amount of F and the corresponding investment fund allocation as well as fundamental factors that influence the investment problem.

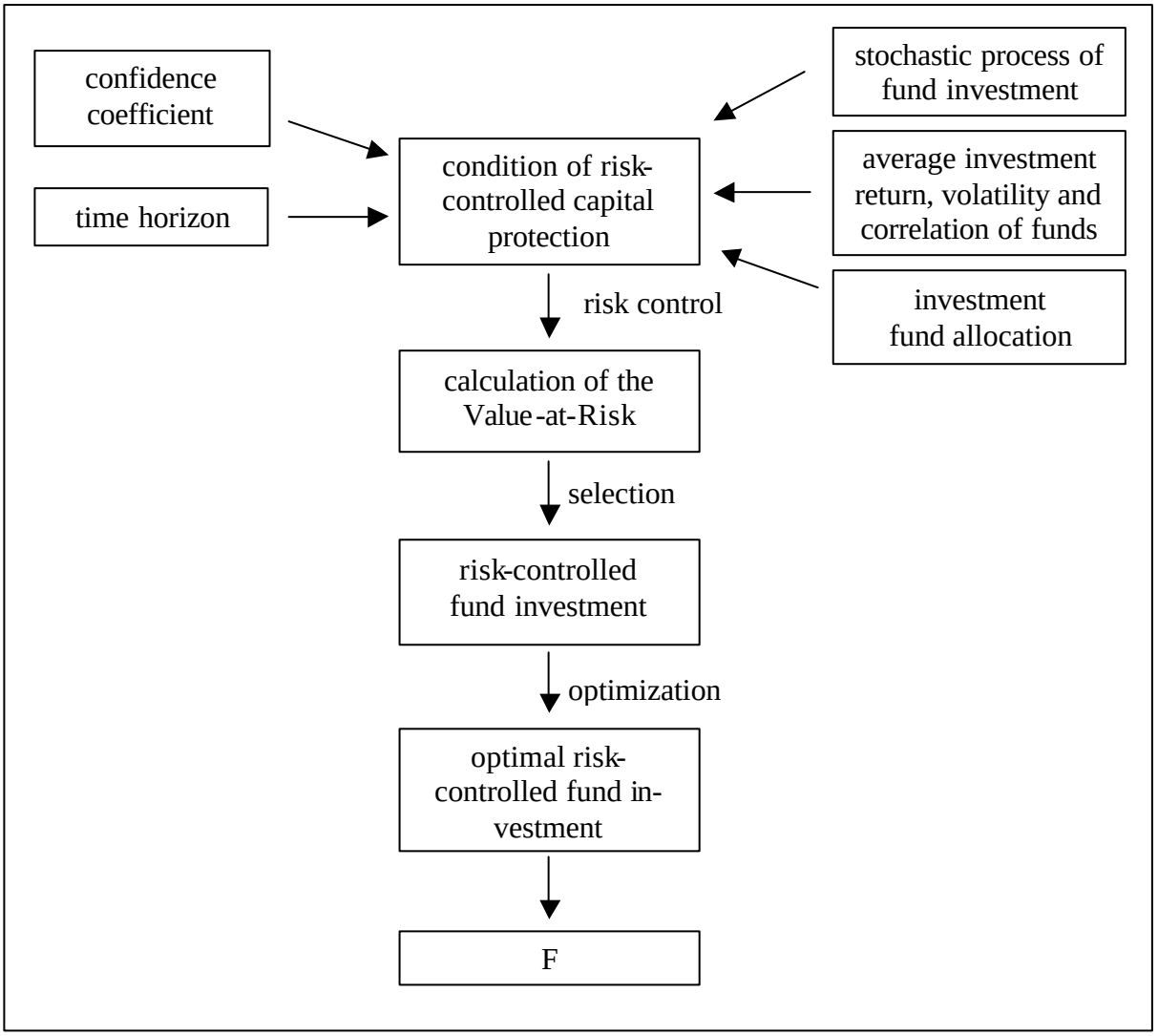

Figure 2: Procedure of our formalization

For further concretion of the general procedure, we limit our analysis to the case of three different investment funds or asset categories.

2.2 Case of three investment funds

For the simultaneous development of three investment funds, e.g. a representative stock, bond or property fund, we assume a multivariate geometric Brownian motion. Since the distribution of F at the end of the time horizon can not be determined in an analytical way, the Value-at-Risk is not analytically definable either and therefore has to be generated in a Monte Carlo-Simulation.

In consequence, the determination of the minimal amount of F and the corresponding optimal investment fund allocation can not be achieved analytically either. Like in Albrecht/Maurer (2002), we use the standard approach of restraining the possible investment fund allocations

to a representative number and vary the investment weights of each fund by steps of 5%, which results in 231 investment fund allocations.

stock funds	0%	0%	…	0%	5%	…	5%	10%	…	10%	…	95%	95%	100%
bond funds	0%	5%	…	100%	0%	…	95%	0%	…	90%	…	0%	5%	0%
property funds	100%	95%	…	0%	95%	…	0%	90%	…	0%	…	5%	0%	0%

Table 1: Representative investment fund allocations

3. Results

The following results refer to the case of three investment funds and are based on the parameters for continuous investment returns in real terms.

average investment returns of stock fund	8%, 5% resp.
average investment return of bond fund	4%
average investment return of property fund	3.3%
volatility of stock fund	25%
volatility of bond fund	6%
volatility of property fund	2%
correlation between stock and bond funds	0.2
correlation between stock and funds	-0.1
correlation between bond and property funds	0.6
issue surcharge of stock fund	5%
issue surcharge of bond fund	3%
issue surcharge of property fund	5%

Table 2: specification of parameters

We use the empirical results for the German market of *Maurer/Schlag* (2002) and *Sebastian* (2003) for our simulations, however, we projected the average returns to be slightly lower and the volatilities to be slightly higher in our prospective model calculations. With regard to the stock fund we take two alternative scenarios into account. On the one hand, we consider an average investment return of 8% in real terms, which often represents the standard estimate of average stock returns for very long time horizons as used in *Pye* (2000). On the other hand, the conservative projection of 5% in real terms serves to obtain information on the sensitivity of the results.

Tables 3 and 4 contain the minimum amounts of F, the corresponding optimal investment fund allocations and the annual annuity dues based on a continuous real money market return of 1.5% according to the respective time horizons as well as the confidence coefficients. From now on we deal with the transformation of the confidence coefficients α into degrees of certainty $(1-\alpha)$. The results refer to average real stock returns of 8% and 5% and assume full capital protection of $(1-h)=1$. As outlined in the appendix, the desired fraction of capital protection does not affect the determination of the optimal investment fund allocation, but merely the amount of F and the annual annuity due.

		time horizon in years				
degree of certainty		5	10	15	20	25
	95%	(5% 0% 95%) 94.851,07 1.060,91	(5% 0% 95%) 81.533,17 1.973,80	(10% 5% 85%) 69.232,59 2.273,47	(10% 5% 85%) 58.189,23 2.401,72	(15% 15% 70%) 48.499,59 2.451,92
	90%	(5% 0% 95%) 93.189,78 1.403,21	(10% 5% 85%) 79.201,37 2.223,04	(15% 20% 65%) 66.248,61 2.493,96	(20% 30% 50%) 54.455,79 2.616,18	(25% 40% 35%) 43.912,82 2.670,29

Table 3: results for an average real stock return of 8%

		time horizon in years				
degree of certainty		5	10	15	20	25
	95%	(5% 0% 95%) 95.552,87 916,31	(5% 0% 95%) 82.774,21 1.841,16	(5% 0% 95%) 71.127,18 2.133,47	(5% 5% 90%) 60.889,03 2.246,63	(5% 5% 90%) 51.978,41 2.286,29
	90%	(5% 0% 95%) 93.888,81 1.259,18	(5% 5% 90%) 80.701,59 2.062,69	(5% 5% 90%) 68.909,11 2.297,37	(5% 10% 85%) 58.607,26 2.377,70	(10% 20% 70%) 49.446,32 2.406,84

Table 4: results for an average real stock return of 5%

All in all, the following plausible dependencies can be observed:

- The longer is the time horizon, the larger is the share invested in stock and bond funds.
- The longer is the time horizon, the smaller is the amount of F, that has to be invested in the risky investment funds and the larger is the amount of MM disposable for the annuity due.
- The higher is the degree of certainty, the lower is the share of stock and bond funds and the larger is the share of property funds as the least risky type of investment fund.

- Using lower average real stock returns leads to consistently larger amounts of F to be invested in risky investment funds and in general to a lower share of stock funds for the optimal investment fund allocation.

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Appendix: Fundamental Methodology

In the general case of N investment funds or asset categories, the development of the value of each fund during n years is determined by

$$1+V_k(n) := \exp\left(\sum_{t=1}^{n} U_k(t)\right), \quad k=1,...,N. \tag{1}$$

Assuming a multivariate geometric Brownian motion, the vectors of continuous real returns $(U_1(t),\cdots,U_k(t),\cdots,U_N(t))$ with $t=1,\cdots,n$ are i.i.d. as

$$(U_1,\cdots,U_k,\cdots,U_N) \sim N(m,\Sigma). \tag{2}$$

The vector of real Log-returns has a multivariate normal distribution.

Given $x=(x_1,\cdots,x_k,\cdots x_N)$ with $0 \le x_k \le 1$ and $\sum x_k = 1$ represents the vector of shares invested in each of the N investment funds or asset categories and $100a_k\,\%$ with $k=1,\cdots,n$ the respective issue surcharges, we obtain the following wealth in real terms per invested unit after a time horizon of n years

$$1+V_n(x) = \sum_{k=1}^{N} x_k \frac{1+V_k(n)}{1+a_k}. \tag{3}$$

The condition of risk-controlled capital protection in real terms is defined as

$$P(F[1+V_n(x)] \le (1-h)W) = 1-\alpha \tag{4}$$

with $0<(1-h)\le 1$ being the desired degree of capital protection. (For example, $(1-h)=0.9$ demonstrates a capital protection in real terms of 90%.)

Given $Q_\alpha(x)$ represents the α-quantile of the random number $1+V_n(x)$, we obtain

$$F = F(x) = \frac{(1-h)W}{Q_\alpha(x)}. \tag{5}$$

Then it is necessary to determine the investment fund allocation x*, that yields

$$F(x) \to \min!.$$

Since equation (5) applies, $F(x)$ is at its minimum, when $Q_\alpha(x)$ is at its maximum. Thus, regardless of the desired degree of capital protection in real terms, from a formal perspective, we merely have to find the investment fund allocation x*, that yields

$$Q_\alpha(x) \to \max!. \tag{6}$$

According to equation (5), $Q_\alpha(x)$ determines the minimum amount of F for a given confidence coefficient α. Capital protection is only feasible, if the amount of F can be financed by the original wealth W, implying $F \leq W$. Because of equation (5), the necessary condition to be fulfilled, is

$$Q_\alpha(x^*) \geq (1-h). \tag{7}$$

Therefore, our analysis can be conducted independent from the amount of original wealth W and based on one unit of wealth instead. For this purpose, it is sufficient to analyze the quantile $Q_\alpha(x)$ and fix the desired degree of capital protection in real terms.

The average development of the value of the optimal fund investment $F(x^*)$ ultimately is

$$E\left[F(x^*)\right] = F\sum_{k=1}^{N} \frac{x_k{}^*}{1+a_k}\, e^{n(m_k+\frac{1}{2}v_k{}^2)}, \tag{8}$$

while $m = (m_k)$, $\Sigma = (v_{kj})$ and $v_k^2 = v_{kk}$.

Finally, we describe the amount of MM in real terms invested in money market funds, that yields the annual annuity due after a time horizon of n years

$$W - F = W - \frac{(1-h)\,W}{Q_\alpha} = \frac{[Q_\alpha - (1-h)]}{Q_\alpha} W. \tag{9}$$

Given i represents the deterministic annual continuous real money market rate, the annual real annuity due is determined by

$$R = \frac{Q_\alpha - (1-h)}{Q_\alpha} W\, q^{n-1}\, \frac{q-1}{q^n - 1} \tag{10}$$

with $q := e^i$. Evidently, the annual real annuity due is positive, if $Q_\alpha > (1-h)$.