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Jörn Barth*

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* Graduiertenkolleg „Allokation auf Güter- und Finanzmärkten“, Universität Mannheim, L13, 15, D-68131 Mannheim, Germany. Tel.: ++49 +621 292-5578, Fax: ++49 +621 292-2789, e-mail: barth@econ.uni-mannheim.de, Fax by e-mail: ++49 ++89 66617-66361. I thank Peter Albrecht, Milan Borkovec, Malte Cherdron, Darrel Duffie, Martin Hellwig, Claudia Klüppelberg, and Marc Windfuhr for stimulating discussions. All errors are my own.

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*Graduiertenkolleg "Allokation auf Güter- und Finanzmärkten", Universität Mannheim, L 13.15, D-68131 Mannheim, Germany. Tel: ++49 +621 292–5578, Fax: ++49 +621 292–2789. e-mail: barth@econ.uni-mannheim.de, Fax by e-mail: ++49 +89 66617–66361. I thank Peter Albrecht, Milan Borkovec, Malte Cherdron, Darrell Duffie, Martin Hellwig, Claudia Klüppelberg, and Marc Windfuhr for stimulating discussions. All errors are my own.

Abstract

A model for the credit risk of a portfolio of market driven financial contracts (for example swaps) is introduced. The viewpoint of the financial institution who holds this portfolio is taken. The default intensity of a single counterparty is assumed to write as a sum of two parts: An *individual* component is unknown and modelled as noise, the *collective* component is known and dependent on market variables (like the interest rate). The influence of the credit events to the market variables is neglected. The advantage of this model is given by the possibility to consider the statistic of the credit events *conditioned* on a market situation. By taking a functional like the expectation value of this conditional statistic only the market variables remain stochastic. Therefore this model is especially suited for measuring the impact of the market variables onto the credit risk.

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0 Introduction

In the last year severe crises in the worldwide financial markets shook the confidence in the management of market and especially credit risk. One of the most prominent examples was given by the LTCM Hedge Fund which got into deep financial struggle because of a misspecification of the models for the credit spreads. This became dangerous in the case of Russian bonds: The quasi-default of the Russian bond issuers lead to the total failure of these models and with this of the risk management of LTCM. In this case it became obvious that the success of risk management is very dependent of the modelling of the risks involved.

In this light even the credit risk of a portfolio of simple bonds appears difficult to handle (We take always the view of the financial institution which holds this portfolio). How to put together the values of the outstanding contracts (i.e. the exposure) which are held by the different counterparties? How to bring in the different default probabilities of the single counterparties? How to take the correlation between default events into consideration? There is no standardized approach to this problem.

With CreditMetrics and CreditRisk+ (see J.P.Morgan (1997) and CreditSuisse (1997)) two theoretical frameworks are presented, which address this problem. While CreditRisk+ is an analytically solvable actuarial approach, CreditMetrics relies on Monte Carlo simulation techniques to estimate the possible losses due to default events. As seen by practioners, both of these models are not easy to implement. For instance a lot of data about correlations between default events of several counterparties is needed to calibrate the so-called "firm-value" approach of CreditMetrics. The assumptions in the CreditRisk+ framework are also not easy to justify from the practical point of view.

The situation is even more difficult if one deals not only with simple contracts like bonds but also with instruments like derivatives whose values depend highly on market variables. Not only the "pure" credit risk of such contracts has to be regarded but also the behaviour of the value of these contracts under changes of these market variables. For example, a swap will have a positive or a negative value in dependency of the changes of the interest rate. The default of the counterparty in the swap contract might therefore lead to no loss or a severe loss depending on the actual value of the interest rate. Neither CreditMetrics nor CreditRisk+ do incorporate these market variables in a natural way. Both of them might be extended to capture market variables, but would lose their analytical tractability or numerical elegance. It is still usual to apply "rules of thumb" in regard to the credit risk of market-driven instruments.

Usually credit risk management models these two "kinds" of risks as independent stochastic variables: Market variables do not influence the credit risk and vice-versa. This "independence assumption" simplifies the analysis of the credit risk of market-driven instruments, because one can treat the different risks separately. This subdivision into "market" and "credit risks" is deeply embodied in the thinking about risk management. This is reflected in the organization of financial institution where mostly one section cares about the credit risk, and another section about the market risk.

But are there really two different kinds of risk? A change of the market situation might also change the financial standing of some counterparties and with this the credit risk of these counterparties. In Duffee (1996a) the independence assumption is discussed with examples of market-driven contracts like swaps. In such contracts which are driven by the change of a market variable and which are subject to credit risk the distinction between market and credit risks becomes uncertain.

As Duffee (1996a) states the independence assumption is difficult to justify even in "normal" market situations. But in extraordinary market situations this assumption is dangerous because the probability that the credit risk of the counterparties changes with the market is very high. And these are the situations risk management should account for. In such extreme situations the failure of the independence assumption is unavoidable. For instance, due to extraordinary changes of the interest rate financial institutions might get into difficulties and the credit risk of these institution will increase dramatically.

However the consideration of dependent credit and market risks is very difficult: In addition to the modelling of the credit risk for each counterparty one has to model also the correlation with the market variables. There are several different possibilities to do this: One could model the value of the assets of the counterparties. This "firm-value" should be dependent of the market variables. Or one models the default intensity as variable with the changes of the market situation. But the defaults could also change the market variables which further complicates the modelling. In realistic cases usually by introducing some kind of correlation the analytical tractability is lost.

But even numerically there are difficulties to take account of the correlation between default und market risks. Given a large portfolio of maybe 10000 counterparties, it is complicated enough to simulate such a great number of counterparties which might default or not, even if the default probabilities are fixed, see Duffie and Singleton (1998). If these default probabilities depend on other factors which have to be simulated too, some of the elegant treatments of the simulation of default times are no more applicable. And the requirements for computing power and time are very large in a "brute force" Monte Carlo simulation approach. If the simulation of the credit risk of a large portfolio has also to be considered over a long time interval, these requierements grow further which the number of time steps involved.

And, at last, even if these models are implemented numerically, it is necessary to estimate the parameters of these models. Proxies for default probabilities might be given by the ratings (for instance of Moody's or Standards & Poor). But there is a lack of reliable data about how these default probabilities are effected due to the correlations to market variables. So the calibration of the models parameters in regard to the interaction between market and credit risks is difficult too.

In this paper a simplified approach is proposed to handle some of the problems mentioned above. The most important feature of this model is the modeling of the credit risk (i.e. the default probability) of a single counterparty as the sum of two components: One component describes the *individual* contribution of the considered counterparty to its credit risk. This *idiosyncratic* risk is not influenced by movements in market variables. In opposite to this the other component describes the change in the default probability due to changes in the market variables, like the interest rate. This component is therefore called the *collective* or *systematic* contribution to the credit risk.

Given an initial rating for one counterparty this approach allows to adjust the default probability to a changed market situation by the systematic compenent. The "official" rating by some rating agency is mostly too "sticky", which means that the rating is adjusted too late, cp. Düllmann et al. (1998). The systematic component gives an instantaneous re-adjusted default probability, which is in agreement with empirical results, cp. Fons (1994).

The crucial point of this model is that the idiosyncratic component is assumed to be unknown to the considered financial institution. It is therefore modeled as "noise". This noise prevents from calculating an effect of a change of the credit risk on the market variables by considering the mathematically reversed relationship. With this the "causality" of this model is determined: The market risks influence the credit risks, but not vice-versa.

This model of a "one-sided" dependency between market and credit risks takes a position between models with "full" dependency (which are far more difficult to handle) and models which incorporate the assumption of the independence. The simplification due to this one-sided dependency is the possibilty of considering the credit events *conditioned* on a fixed market situation. With the market variables fixed, only the stochasticity of the individual risks remains[1]. In this situation one can take the conditioned expectation value (or another functional as the upper percentile) of the individual risks to obtain the expected loss of a portfolio due to defaults *given* a market situation[2]. By varying this market situation one can use the model for measuring the impact of this movements in market variables onto the credit risk: Only a simulation of the market variables has to be performed to determine the conditioned expected loss due to credit events, no simulation of the default processes is required.

In this model the influence of the credit events on the market is neglected. This may be a severe neglection if for example due to the default of a larger amount of counterparties the spread on bonds widens in the whole market as it was the case in the Russian crises mentioned at the beginning of this paper. But this will primarily effect bond prices, the effects on prices of market-driven instruments as swaps for example might be much smaller.

This model is suited for different applications. The determination of the risk-based capital is maybe the most important one and discussed in detail with several simulation studies in Barth (1999). Another application would be the calculation of spreads for different sorts of contracts. The emphasis should lay on the tracking of the market variables.

In the first chapter the model is presented and discussed. One of the most important features is the modelling of the default risk. A "response function" of the credit risk due to the changes in the market risks is introduced. The modelling of this response function and its calibration with empirical data is discussed in the second chapter.

1 The Loss Process

1.1 Definition

We consider a fixed number of N counterparties $a = 1, \dots, N$ in the time interval $[0, T]$. The netted value of the contracts which are traded with counterparty a is denoted by V_a. We assume netting of all the contracts with one counterparty, but netting is not applied between different counterparties. We call $V_a^+(t)$ the exposure to counterparty a at time

[1] An investigation of a similar situation in a micro-economic context is given in Berninghaus (1977).

[2] If there is a large number of individual risk, an application of the Law of Large Numbers might appear as appropriate to eliminate the indiviual risks instead of taking the expectation value. This will be reported in a later version of this paper.

$t \in [0, T]$, where $(\cdot)^+ = \max(0, \cdot)$. It should not be possible that the considered financial institution achieves a gain through the default of a counterparty. The consideration of the exposure alone might be suited for the setting of risk limits, but not for a quantitative analysis of the credit risk of a portfolio.

The event of a default of one counterparty a is modelled as each jump of a Poisson process N_a with stochastic intensity λ_a. The stochastic intensity λ_a is specified in the next section. Here we take λ_a as given and concentrate on the modelling of the default events and the associated losses.

We introduce the default indicator process dN_a as

$$dN_a(t) = \lim_{\Delta t \to 0} \left(N_a(t) \Leftrightarrow N_a(t \Leftrightarrow \Delta t) \right). \tag{1}$$

Let $\{\tau_{a,i}\}_{i=1,\ldots,M_a}$ be the set of jump times of the Poisson process N_a in the time interval $[0, T]$, where M_a is the number of the jump times of counterparty a in $[0, T]$. Then dN_a can be rewritten as

$$dN_a(t) = \lim_{\Delta t \to 0} 1_{\{\tau_{a,i} \in [t - \Delta t, t)\}} \tag{2}$$

for one $i = 1, \ldots, M_a$.

With the modelling of the default event as *each* jump of N_a it follows that counterparty a might default more than one time in $[0, T]$, i.e. $M_a > 1$. This approximation is usually done in many applications in risk management and is known as the "Poisson approximation", cf. for example CreditSuisse (1997). The probability for $M_a > 1$ is in quadratic order of the total default probility of counterparty a in $[0, T]$ and therefore very small for realistic parameters in credit-risk management. Without this approximation the default time has to be regarded as a first-passage time of N_a, which is analytically difficult to handle.

With these remarks[3] the discounted credit loss $dL(t)$ in the time interval $[t \Leftrightarrow dt, t)$ writes as (cp. for a similar approach Albrecht et al. (1996)):

$$dL(t) = \frac{1}{B(0, t)} \sum_{a=1}^{N} V_a^+(t) dN_a(t), \tag{3}$$

where $B(t_1, t_2)$ is the money market account, i.e.

$$B(t_1, t_2) = \exp\left[\int_{t_1}^{t_2} r(s) ds \right], \tag{4}$$

[3] We only consider the total loss of the value of the exposure in the default case. The model might be easily extended to incorporate loss fractions smaller than 1.

with $r(t)$ as the riskless interest rate at time $t \in [0, T]$. The loss in the whole time interval $[0, T]$ is obtained by

$$L(t) = \int_0^t dL, \tag{5}$$

where the integral is defined for each realisation of dN_a (a reference for such random measures can be found in Daley and Vere-Jones (1988)). $L(t)$ can be rewritten as

$$L(t) = \sum_{a=1}^{N} \sum_{i=1}^{M_a} \frac{1}{B(0, \tau_{a,i})} V_a^+(\tau_{a,i}), \tag{6}$$

where the characteristic function eq.(2) is used with the jump times $\{\tau_{a,i}\}_{i=1,\ldots,M_a}$.

With dt infinitesimal the expectation value of dN_a can be written as

$$\begin{aligned} \mathrm{E}\left[dN_a(t)\right] &= \mathrm{Prob}\left\{dN_a(t) = 1\right\} \\ &= \mathrm{Prob}\left\{\tau_a \in [t \Leftrightarrow dt, t)\right\} \\ &= \mathrm{Prob}\left\{N_a(t) = 1 | N_a(t \Leftrightarrow dt) = 0\right\} \\ &= \lambda_a(t) dt. \end{aligned} \tag{7}$$

The expression $\lambda_a(t)dt$ gives the probability that the default event takes place in the time interval $[t \Leftrightarrow dt, t)$.

Another modelling approach would be to consider the (unconditioned) Poisson process $N_a(t)$ instead of $dN_a(t)$ in eq.(3). We do not consider this possiblity in this paper, because in regard to the exposure V_a at time t the probability of default in a small time interval prior to t is important, not the probability of default in $[0, t]$.

1.2 Modelling the Risks

In most of the models which deal with the credit risk of market-driven contracts the default process and the market risk process are assumed to be independent. But this assumption is difficult to justify, cf. Duffee (1996a), Hull (1989). However it is difficult to overcome this assumption, because there is only little information and data for estimating a correlation between market and credit risks. In this section we describe one possibility for incorporating this correlation. Here we describe only the theoretical model, for matters of calibration and estimation of this model we refer to chapter 2.

We model a "one-sided dependency": The market variables should be "independent" of the credit risks, but the credit risks could be influenced by the market variables. For example, a rise in the interest rate might worsen the credit qualtity of many firms, but a single "downgrading" does not influence the interest rate[4].

To do this, we distinguish two "sorts" of risks: First there are "systematic risks" like market variables, which are known to everybody. In chapter 2 we regard only the interest rate r as a systematic risk, here r can be a vector of market variables which are assumed to be important in the application of the model. Second there are "individual risks", which are not common knowledge: In the following we regard as the individual risk of counterparty a the financial state, which is not known exactly to the considered financial institution. We model this uncertainty about this financial state at time t as a random variable $\epsilon_a(t)$ with a known distribution, see below.

Further we model the credit risks as a function of the market risk r and the residual individual component ϵ_a. Here we consider as the credit risk of counterparty a the intensity λ_a of the default process I_a, which we assume has the following form:

$$\lambda_a(t) = S_a(r(t), t) + \epsilon_a(t), \tag{8}$$

with a well-behaved function $S_a : \mathrm{R}^+ \times \mathrm{R} \to [l_a, \infty)$, which remains to be modelled, see chapter 2. $S_a(r(t), t)$ is the *known* default intensity of counterparty a in the market situation $r(t)$. We will refer to S_a as the "response function", because it describes the changes in the credit risks due to changes in the market variables r. The constant $l_a > 0$ gives a lower bound for this intensity. It is one of the fundamental assumptions in this paper that this function is only dependent on the actual value of r, not on the path of r up to this time. The residual $\epsilon_a : \mathrm{R}^+ \to [-l_a, +l_a]$ models the difference of the "true" default intensity λ_a and the official rating $S_a(r(t), t)$, which is given for instance by Moody's. S_a describes the spread in the interest rate due to the known default risk of counterparty a against a riskless interest rate. This point will be getting more clear in the next section.

We model the individual risk ϵ_a as diffusion process without any drift. With this assumption the doubly stochastic process N_a based on eq.(8) is well-defined, cf. Grandell (1976). One important condition is the independence of ϵ_a of the market-parameter processes r. Moreover the individual risk processes $\{\epsilon_a\}_{a=1,\ldots,N}$ should be pairwise independent for all $a \neq a'$.

[4] There will be different opinions if this approximation works very well in the case of a worst case scenario (which will be studied in fact), because in worst case scenarios there might be influences of the defaults on the level of the interest rate. But we think, that this approximation works better than the "traditional" independence assumption between market und default risks. Even numerically a model which incorporate the "full" dependency will be much more computing-intensiv.

Further we assume, that the drift and the volatility function of the diffusion process ϵ_a is "common knowledge", i.e. the marginal distributions of $\epsilon_a(t)$ for all $t \in [0, T]$ are known to the considered financial institution. We state several conditions on these distributions: The assumption, that the processes of the ϵ_a are without drift results in $\mathrm{E}[\epsilon_a(t)] = 0$ for all a and t. Without this assumption there would be an expected difference between the known spread S_a and the default intensity λ_a at some time $t \in [0, T]$. Such an expected difference would be incorporated to the spread S_a by the rating agency. Besides it should be excluded that $\lambda_a < 0$: Therefore we restrict ϵ_a to the interval $[\Leftrightarrow l_a, l_a]$ by cutting off values which are outside of the interval[5].

1.3 Treatment of the Exposure Process

The application of a straightforward pricing model for calculating the exposure V_a neglects the fundamental information asymmetry, which is responsible for the modelling of the default process as a Poisson process, cf. Duffee (1996a). Therefore the pricing models have to be modified for the use in a "world with default". One approach is given by the work of Duffie and Singleton (1995): The riskless interest rate r is replaced by a modified interest rate[6] $R_a(t) = r(t) + \lambda_a^Q(t)$ under the equivalent martingale measure Q. The spread λ_a^Q is given by the default intensity under the equivalent martingale measure Q of the considered counterparty. Then the exposure writes as[7]:

$$V_a(t) = \mathrm{E}^Q \left[\exp\left(\Leftrightarrow \int_t^T R_a(s) ds \right) Z(r(T)) | \mathcal{F}_t \right]. \tag{11}$$

[5] We try to keep the notation simple. To be strict, one has to write instead of eq.(8)

$$\lambda_a(t) = S_a(r(t), t) + C_a(\epsilon_a(t)), \tag{9}$$

with a cut-off function

$$C_a : \mathrm{R} \to [-l_a, l_a], \quad \epsilon_a(t) \mapsto \begin{cases} l_a & \text{if } \epsilon_a(t) > l_a, \\ \epsilon_a(t) & \text{if } \epsilon_a(t) \in [-l_a, l_a], \\ -l_a & \text{if } \epsilon_a(t) < -l_a. \end{cases} \tag{10}$$

Because of its symmetry this function C_a does not affect the property $\mathrm{E}[C_a(\epsilon_a(t))] = 0$, if the marginal distributions of the process ϵ_a are symmetric themselves.

[6] If a loss rate $L_a(t) < 1$ is considered, one has to write for the modified interest rate $R_a(t) = r(t) + L_a(t)\lambda_a^Q(t)$.

[7] This formula is valid only if there is one contract with counterparty a which pays $Z(r(T))$ at time T, otherwise it must be summed over the different contracts und times of payoffs.

Using eq.(8) and separating the expectation value because of the independence of r and ϵ_a:

$$\begin{aligned} V_a(t) &= \mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \left(r(s)+S_a^Q(r(s),s)\right)ds\right) Z(r(T))|\mathcal{F}_t\right] \times \\ & \qquad \mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \epsilon_a^Q(s)ds\right)|\mathcal{F}_t\right]. \end{aligned} \tag{12}$$

The filtration $\mathcal{F}$ describes the development of the information of the considered financial institution.

With the expression

$$\mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \epsilon_a^Q(s)ds\right)|\mathcal{F}_t\right] \tag{13}$$

the stochastic part of the individual risks is eliminated from the pricing formula. Expression eq.(13) could be evaluated because the distribution of the ϵ and the transformation of the real measure to the equivalent martingale measure Q is known[8]. By defining

$$f_a^Q(t,T) = \Leftrightarrow\ln \mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \epsilon_a^Q(s)ds\right)|\mathcal{F}_t\right] \tag{14}$$

one could write[9]

$$\mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \epsilon_a^Q(s)ds\right)|\mathcal{F}_t\right] = \exp\left(\Leftrightarrow f_a^Q(t,T)\right). \tag{16}$$

By inserting eq.(16) into eq.(12)

$$V_a(t) = \exp\left(\Leftrightarrow f_a^Q(t,T)\right)\mathrm{E}^Q\left[\exp\left(\Leftrightarrow\int_t^T \left(r(s)+S_a^Q(r(s),s)\right)ds\right) X_T|\mathcal{F}_t\right], \tag{17}$$

the term $f_a^Q(t,T)$ is identified as an additional contribution to the spread S^Q, which can be interpreted as a premium, which has to be payed because of the uncertainty about the individual risk of the counterparty a in the time interval $t \in [0,T]$.

---

[8] The switching between the intensity under the real measure S_a and the intensity under the equivalent martingale measure S_a^Q can be done by the Girsanov transformation for Poisson processes, but empirically established connections exists too, cf. Fons (1994).

[9] It is maybe convenient to approximate eq.(16) with a zero-order approximation (so that f_a^Q has not to take into account), but this is not necessary here:

$$\mathrm{E}^Q\left[\exp\left(-\int_t^T \epsilon_a^Q(s)ds\right)|\mathcal{F}_t\right] \approx 1. \tag{15}$$

This approximation is equivalent to the neglection of the influence of the individual risks ϵ_a to the pricing of the contract.

The most important point of this calculation is that the stochasticity of the individual default risks does not influence directly the pricing of the contracts, which makes sense from the economic point of view, because only the known quantities S_a and f_a should enter into the valuation formula. In a practical implementation one would use instead empirically determined spreads which also account for other determinants as liquidity and sector effects not included in this theoretical approach.

1.4 Treatment of the Default Process

dN_a describes the indicator of a jump of a Poisson process N_a with a stochastic intensity $\lambda_a(t) = S_a(r(t), t) + \epsilon_a(t)$, i.e. a doubly stochastic process or Cox process. This process is difficult to handle both analytically and numerically. But this process is responsible for the "weights" of the different exposures of the counterparties in regard to the credit risk of the entire portfolio. Therefore it is necessary to obtain an expression which is easier to handle.

In the approach of CreditMetrics the market risk is taken into account by regarding only the average exposure or the maximum exposure. The random variable $V_a^+(r(t), t)$ is replaced by an expected value (as the average) or an upper percentile (as the maximum) over the time interval $t \in [0, T]$. This eliminates the dependency of the market stochasticity which is given by r. Then this average or maximum exposure is regarded as a fixed quantity and treated like the nominal value of a bond.

In contrast to the approach of CreditMetrics we concentrate on the effects of the market variables on the credit risks by using the response function $S_a(r)$ in eq.(8). We take an expectation value of the *credit* risks instead of the *market* risks. This expectation value is conditioned on the market risk situation which is in this context given by the interest rate r. This approach is the opposite to the approach of CreditMetrics with respect to the treatment of the credit and market risks: The individual components of the credit risk ϵ_a are eliminated by taking the expectation value, while only the market-driven collective component $S_a(r)$ of the credit risk remains.

By taking the expectation value of eq.(3), conditioned on a fixed market scenario r

$$\mathrm{E}\left[dL(t)|r(t)\right] = \mathrm{E}\left[\sum_{a=1}^{N} V_a^+(t) dN_a(t)|r(t)\right] \tag{18}$$

and by calculating (for ease of notation we suppress the dependency of r)

$$\mathrm{E}\left[V_a^+(t) dN_a(t)|r(t)\right]$$

$$
\begin{aligned}
&= V_a^+(t)\mathrm{E}\left[dN_a(t)|r(t)\right] \\
&= V_a^+(t)\mathrm{E}\left[\mathrm{E}\left[dN_a(t)|\epsilon_a(t)\right]|r(t)\right] \\
&= V_a^+(t)\mathrm{E}\left[\lambda_a(t)dt|r(t)\right] \\
&= V_a^+(t)\left[S_a(r(t),t)dt + \mathrm{E}\left[\epsilon_a(t)\right]\right] \\
&= V_a^+(t)S_a(r(t),t)dt, \qquad (19)
\end{aligned}
$$

the default process is eliminated. Only the market risk process remains as a source of uncertainty.

If one is interested in analyzing the impact of the market variables on the credit risk, one might consider the conditional expectation value of the loss process, which writes as

$$
\mathrm{E}\left[dL(t)|r(t)\right] = \sum_{a=1}^{N} V_a^+(t)S_a(r(t),t)dt \qquad (20)
$$

instead of examining the loss process eq.(3) itself. The "weights" $S_a(r(t),t)dt$ represent the known probability of default in the time interval $[t \Leftrightarrow dt, t)$. This expression is much easier to handle in comparison to eq.(3), because only the stochasticity of r remains to be considered (analytically or numerically).

In addition to the expectation value the variance can be considered. The conditional variance of the default process of the single counterparty a is calculated as

$$
\begin{aligned}
&\mathrm{Var}\left[V_a^+(t)dN_a(t)|r(t)\right] \\
&= (V_a^+(t))^2\left[S_a(r)dt \Leftrightarrow S_a(r)^2dt^2\right] \\
&= (V_a^+(t))^2\left[S_a(r)dt\right]. \qquad (21)
\end{aligned}
$$

The conditional variance of the total portfolio writes as the sum of the single variances, because the default processes I_a are independent for $a = 1, \ldots, N$ conditioned on a market scenario $r(t)$:

$$
\begin{aligned}
&\mathrm{Var}\left[\sum_{a=1}^{N} V_a^+(t)dN_a(t)|r(t)\right] \\
&= \sum_{a=1}^{N}(V_a^+(t))^2\left[S_a(r)dt \Leftrightarrow S_a(r)^2dt^2\right] \\
&= \sum_{a=1}^{N}(V_a^+(t))^2\left[S_a(r)dt\right]. \qquad (22)
\end{aligned}
$$

It depends of the kind of application of this model in which sense the variance has to be regarded. Again the important point is the elimination of the individual default characteristic ϵ_a.

In the case of a large portfolio one might think about the Law of Large Numbers to eliminate the individual default stochastics. This will lead to the same result eq.(20), but the interpretation would be a stronger one: Instead of seeing eq.(20) as the *expected* loss conditioned in a given market scenario, eq.(20) would be the *real* loss in this market situation. No variance has to be regarded. Compare also to the application of the Law of Large Numbers in the "large economy" in Berninghaus (1977), where a similar distinction of risks is introduced and conditional arguments are used the same way. But unfortunately the Law of Large Numbers is not applicable because in eq.(3) an *addition* of the indiviual risks is considered instead of a *subdivision*, i.e. the factor $1/N$ is missing. The application of the Law of Large Numbers is not valid in this case.

But investigations of Nielsen (1985) and Hellwig (1995) justify the use of large number arguments for the addition of individual risks. The most striking result of these investigations is that even risk-averse agents do not consider the growing variance which arise by the addition of risks and take only the expectation value of the sum of the risks into account for their decisions (under mild restrictions on the utility functions of the agents). The application of these results in regard to the problems in this paper are currently under way und will be reported in a later version of this paper.

1.5 Comparison to the Approaches of CreditMetrics and CreditRisk+

In this section a short comparison of the model described here and the framework of CreditMetrics (J.P.Morgan (1997)) and CreditRisk+ (CreditSuisse (1997)) is given. A useful comparison of the structural similarities and differences of CreditMetrics and CreditRisk+ is given in Gordy (1998).

CreditMetrics is a so-called "firm value" model: A latent variable y_a describes the value of the assets of a counterparty a. If y_a drops under a certain lower barrier d_a (which might depend on the rating of the firm), the counterparty defaults. In opposite to this approach the model decribed in this paper is an "intensity-based" model: With eq.(8) the default intensities are modeled rather than the firm value. In Gordy (1998) it is shown that the firm value approach of CreditMetrics can be transformed into an intensity approach, but the resulting intensity is (with fixed market parameters) determinstic, while the intensity eq.(8) is stochastic due to the noise of the individual credit risks.

But in CreditMetrics the structure of the latent variable y_a is modeled similar to the model described here: As Gordy (1998) notes, y_a writes as a sum of weighted random risk factors plus an individual noise term. The interpretation of these terms is similar to the interpretation of these terms in the model of this paper. The risk factors in CreditMetrics could be stock indices of different industry sectors and the weights of these risk factors give the composition of the dependency of the considered counterparties of these sectors. In opposite to this we model a response function to market variables (as the interest rate). This response function could have a more complicated structure than the linear structure of a weighted sum. Mostly the risk-factors in CreditMetrics are assumed to be normally distributed while in this model the response function maps the actual value of the market variables.

The firm-value approach in CreditMetrics allows for multi-state outcomes (as it is needed to model rating migration), an intensity approach only allows the events "no default" or "default". This is the case with the model described here and the approach of CreditRisk+. Both are intensity-based models; multi-state outcomes are not possible.

CreditRisk+ and the model of this paper are similar in that the correlation between defaults of several counterparties are triggered by some underlying variable. Further the idea of modelling the intensity function based on underlying risk factors is a common feature. In CreditRisk+ these risk factors are weighted and summed and then multiplied with the initial default probability, which might be given by the rating of the considered counterparty. This is a special kind of a response function. The intensity function in the model described in this paper is modeled similarly: Instead of the weighted sum the risk factors are mapped by a more general response function S_a. This response function will be gauged so that the initial configuration of the risk factors give the initial default probability induced by the official rating. Beyond that a noise term is added to model the individual risk compenent, so that a stochastic volatility results. In this model the noisy term is important for the economic "causality" of market and credit risks. This noisy term is not incorporated in the model of CreditRisk+, where only the systematic component is regarded[10].

There are some further differences: CreditRisk+ is trimmed to be analytically solvable. To succeed it is necessary to assume, that the risk factors are Gamma distributed. Further the exposures and losses have to be multiplicates of a standard unit of exposure or loss. In CreditRisk+ probability generating functions are considered to give analytical solu-

[10] In CreditRisk+ only bonds instead of market-driven instruments are regarded. There is no common factor which drives the default probability and the exposure (as it is the case in the model described in this paper). Because of this there is a trivial independence of the exposures and the market variables. Therefore no modelling of the "causal" relationship between these risks is required.

tions. These assumptions are not made in the model of this paper, because the analytical tractability was not the primary objective.

Instead the model described here is trimmed to follow the changes of the market variables over a given time interval. This is not the case for both CreditRisk+ and CreditMetrics: Both concentrate on the credit risks in a fixed market situation (which might be the best way if only bonds are regarded), while the model of this paper concentrates on the effects of the market variables by "eliminating" the individual credit risk components by taking the expectation value conditioned on the market. But the modelling of the response function S_a has not yet be discussed, this will be done in the next chapter.

2 Modelling the Correlation

In chapter 1, a model was introduced which captures effects of the correlation between market variables and the default intensity. In this chapter, we specialize to the consideration of the interest rate r as a market variable. We assume that there are no other market variables than r which are important to consider. Certainly in every realistic situation there will other variables. But here we will try to calibrate the model described in chapter 1 in the simplest case. An extension to other market variables is straightforward, if emprical data is available.

We first review some related economic topics which concern interactions between changes in the interest rate and default probabilities. There exist some theoretical models where such an interaction might be included. These are reviewed shortly thereafter. After that several possibilities to model such a correlation in terms of the response function S_a (see eq.(8)) are proposed. Empirical evidence for choosing the parameters of this response function is discussed at the end of this chapter.

2.1 Economic Surroundings

There are many ways in which a change in the term structure of interest rates could influence the financial standing and with this the default probability of a firm (which might be one counterparty in the considered portfolio). For example, a rise of the short rate leads directly to a rise of the costs of short rate debt. Firms which are exposed to short rate debt are in this situation more likely to get into financial difficulties. In this case $S_a(r)$ (see eq.(20), we drop the explicit time dependency in the notation) will be an increasing function in $r(t)$ with $S_a(r) \to \infty$ for $r \to \infty$. For $r \to 0$ one would expect $S_a(r) \to c_a$ with $c_a > 0$ a constant, maybe dependent on a. One could interpret this constant c_a as the default intensity of a which is independent of the interest rate.

As an example for a more indirect influence Estrella and Hardouvelis (1991) give empirical evidence, that a positive slope of the yield curve is a predictor of a future increase of economic activity. This growth in economic activity will strengthen the financial situation of some firms which induces a lower default probability. A flattening of the yield curve predicts falling economic activity. On average this reduces the financial standing of individual firms.

These examples illustrate the problems of modelling the correlation between the interest rate and the default probability: First, not only the single firm is influenced directly, but also the economic surroundings which will influence this firm indirectly again. Second, different firms might react differently to changes in the interest rate. A third problem is given by the difficulties to estimate the default probability which are reviewed later.

2.2 Theoretical Models

There are some theoretical models for pricing the credit risk which allows for a correlation of the default probability with the interest rate. In opposite to these models we am not interested primarily in pricing this risk. We give a very short and incomplete survey of some of these models.

Cooper and Mello (1991) provide a firm value model for pricing the default risk of currency and interest rate swaps. The process of the value of the firm and the interest rate process are correlated. The swap rates are related to debt market spreads. They find that the swap spread is inversely proportional to the correlation parameter. In the case of risky bonds Longstaff and Schwartz (1995) present another firm value model. They find that the credit spread is influenced by the correlation between the assets of the firm and the interest rate. This might give an explanation for the different yield rates between firms of different industry sectors with the same credit rating. Another theoretic finding is that the credit spreads are negatively correlated to the level of the interest rate.

In opposite to the firm value models there are "reduced form" models, which regard an exogenously given hazard rate process instead of a firm value process. Duffie and Singleton (1995) value risky bonds in such a model where the hazard rate process might depend on market variables. The yield spread due to the default risk is given by the intensity of the hazard rate process times the loss rate, but there might be other determinants of the yield spread. The advantage of this approach is that the well-known pricing models without default risks remain structurally the same, only the interest rate is modified under the equivalent martingale measure (see section 1.3). In Duffie and Huang (1996) this model is extended to value the credit risk of swaps. Again the default characteristics are allowed to be influenced by market parameters.

2.3 Response Functions

We will describe the technique and the results of several simulations of a portfolio of interest rate swaps in Barth (1999), where the (known) default intensities S_a are dependent of the short rate r. Several different response functions S_a are implemented. Though the form of this function is somewhat arbitrary (and might depend on the specific counterparty) there is some empirical evidence which might give hints not for choosing the functional form of S_a but for choosing the parameters if such a function is proposed. In the following we will discuss several functions S_a before listing some empirical results.

To be consistent with the model presented in chapter 2, We have to assume that only the actual value of r is responsible for the default intensity S_a, i.e. there should be no path dependency. Therefore the same $S_a(r(t), t)$ results regardless if there is a slow rise of the interest rate r in $[0, t]$ or a sudden increase (followed by a calm period for instance). $S_a(r(0))$ is given by the rating of a at the starting time $t = 0$.

As the first possibility we modify the function used by Hull (1989)[11] and obtain

$$S_a^e(r(t)) = S_a^e(r(0)) \exp\left[k_a \left(r(t) \Leftrightarrow r(0)\right)\right], \tag{23}$$

with a counterparty-specific constant k_a and a starting value $S_a(r(0))$ for the known default intensity. We will call k_a the response coefficient. This function is displayed in figure 1. Hull uses a time-averaged r instead of $r(t)$ which leads to a path dependency. Further he introduces an explicit time dependency by multiplying the exponent by t, which results in a dependency on the strength of the correlation given a fixed k on the chosen time period. This is not the case in our application. We avoid this explicit time and path dependency. This has the advantage of a multiplicative structure concerning the intensities. Only the parameter k_a is left to be chosen.

In the first order approximation of eq.(23) the interpretation of k_a is straightforward:

$$S_a^e(r(t)) = S_a^e(r(0)) \left[1 + k_a \left(r(t) \Leftrightarrow r(0)\right)\right] + \mathcal{O}\left(\left(r(t) \Leftrightarrow r(0)\right)^2\right). \tag{24}$$

[11] To our knowledge the only other investigation with an explicit modelling of such a function.

Given a change of one percent in the interest rate $r(t) \Leftrightarrow r(0) = 0.01$, the default intensity changes by k_a percent of the initial value $S_a^e(r(0))$.

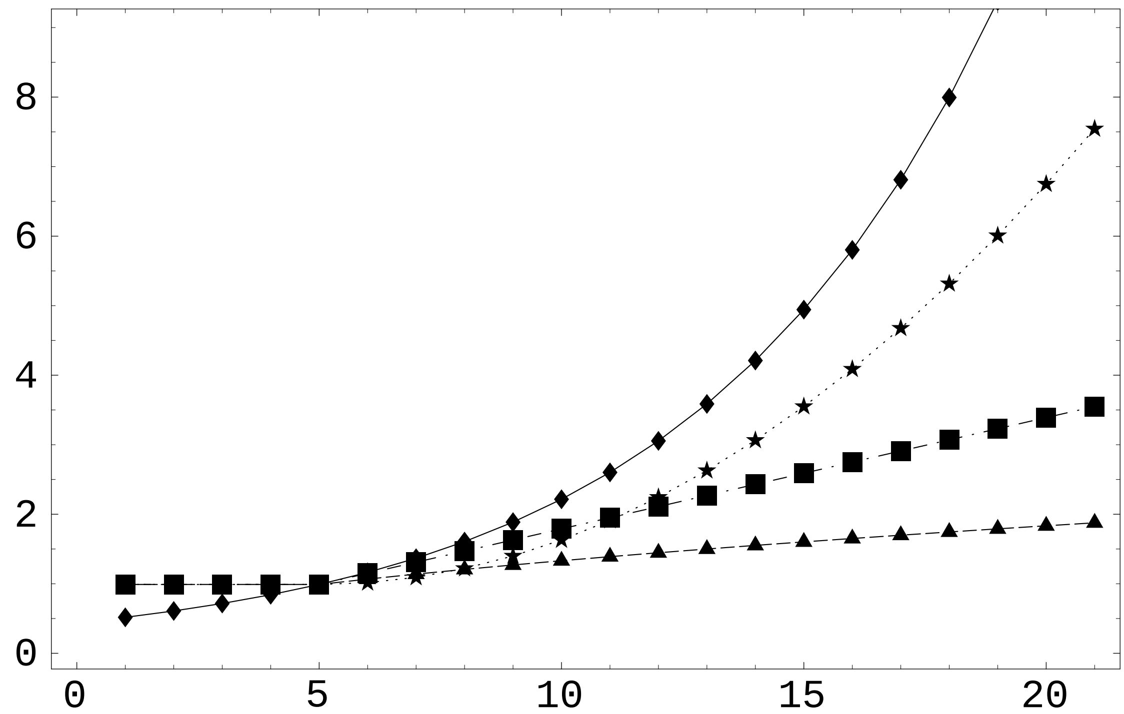

Figure 1: *Correlation functions for $k_a = 16$ and $r(0) = 0.05$ and $S_a(r(0)) = 1$: Exponential eq.(23) (checks), quadratic eq.(25) (stars), linear eq.(26) (squares), and square-root eq.(27) (triangles). The value of the correlation function S is plotted versus r.*

If one regards one realisation of the short rate process, the resulting "paths in default intensity" $S_a(r(t), t)$ might be compared with the empirically tracked one year default probabilities in Fons (1994): They look nearly the same, if one is willing to believe that the only driving force for a change in the default probability is the interest rate (as it is assumed in this paper): For companies which are rated investment-grade these paths are nearly flat at a fixed level. This situation corresponds to a very small k_a in eq.(23). For companies with speculative-grade rating these paths are much more volatile and correspond therefore to a high k_a.

Other function S_a which we use in Barth (1999) are quadratic, linear, and square-root functions (slightly modified not to be symmetric or hit 0) and displayed in figure 1:

$$S_a^q(r(t)) = S_a^q(r(0))\left(1 + \max\left\{0, \operatorname{sgn}\left(k_a(r(t) \Leftrightarrow r(0))\right)\left(k_a(r(t) \Leftrightarrow r(0))\right)^2\right\}\right), \quad (25)$$

$$S_a^l(r(t)) = S_a^l(r(0)) \max\left\{1, 1 + k_a\left(r(t) \Leftrightarrow r(0)\right)\right\}, \quad (26)$$

$$S_a^r(r(t)) = S_a^r(r(0))\sqrt{\max\left\{1, 1 + k_a\left(r(t) \Leftrightarrow r(0)\right)\right\}}. \quad (27)$$

In all these functions only the unfavourable market move $k_a(r(t) \Leftrightarrow r(0)) > 0$ (as seen from the counterparty) leads to a larger default probability. In the "good" case $k_a(r(t) \Leftrightarrow r(0)) < 0$ the default probability does not change. These functions are introduced to exclude the possibility that some results of the numerical investigation are driven only by the fast exponential growth of the correlation function eq.(23).

2.4 Empirical Estimations

The credit rating alone is not a very reliable measure for the default probability. In many cases the yield spread of corporate bonds issued by the regarded firm will give a better estimate: Duffie and Singleton (1995) have shown one example for the variations of the yield rate of a firm with a constant credit rating. The yield spread of corporate bonds to treasury yields is determined partly by the expected default probability of the considered firm. There are also other determinants for the magnitude of the yield spread, for example liquidity and industry specific effects. Duffee (1996b) reports that firms which belong to different industry sectors show different yield spreads even when they are rated equally. It would be necessary to eliminate these other determinants to get an unbiased estimate of the default probability based on the yield spread.

To our knowledge there are only three empirical investigations of the correlation between the interest rate and the spread (which we regard as a proxy for the default probability): Longstaff and Schwartz (1995), Duffee (1996b), and Düllmann et al. (1998). All three consider the yield spread between corporate and treasury bonds as a proxy for the default risk. By accepting this proxy we assume with the existence of S_a that the term structure of yield spread is flat, because S_a does not depend on the maturity. This is a good approximation for longer maturities. Longstaff and Schwartz (1995) and Duffee (1996b) consider the US bond market while Düllmann et al. (1998) consider Deutschemark-denominated bonds.

All three regress the change in the yield spread on the change in the interest rate. Duffee (1996b) and Düllmann et al. (1998) add a proxy for the slope of the term structure. Furthermore Duffee (1996b) and Düllmann et al. (1998) are more careful with eliminating other determinants of the yield spread (for example tax and liquidity effects) in order to provide a measure for the default risk alone. There might be a problem with these regressions if the assumed causality is called in question.

We will refer only to Duffee (1996b) and we will describe only the very outlines of this work. On the basis of monthly data he estimates the coefficient b_1 of the following equation:

$$\Delta Y_t = b_0 + b_1 \Delta r_t + \ldots + \epsilon. \tag{28}$$

with ΔY the change in the spread of treasury yields and corporate bonds and Δr the monthly change in the three month bill yield (which we see as a proxy for the short rate). There is one other term in eq.(28) which takes account of the slope of the term structure, but we am only interested in the absolute level[12]. These regressions are run for different industry sectors. Duffee tries to eliminate other determinants of the yield spread than the market price of default risk. He admits that if there was any other source than the default risk which influences the corporate spreads systematically his results will be biased.

The main finding of these regression is a strong negative correlation. This finding is in agreement with the theoretical and empirical result of Longstaff and Schwartz (1995) and Düllmann et al. (1998). In table 1 we list only the estimated b_1 for "long maturity" bonds (15-30 years).

Rating	Aaa	Aa	A	Baa
b_1	-0.112	-0.155	-0.194	-0.338

Table 1: *Results of Duffee (1996b) for the regression coefficient b_1, estimated for long maturity bonds.*

These results may be interpreted as follows: A yield increase of the corporate bonds due to a rise of the short rate by 100 basispoints is b_1 times 100 basispoints less than the increase of the treasury yield. In other words, the spread ΔY is diminished by b_1 times 100 bp. This is interpreted as that rising interest rates are related to a growth in economic activity and therefore a default of the individual firms become more remote. This result is contrary to the example at the beginning of this chapter. Another observation is that the correlation tends to rise as the rating falls.

These results refer to an average value in different industry sectors (we have reported only the average statistic over these sectors) and not to single firm values. Do we have to take individual firm data for an application of the model described in this paper? By orientating our model at these averaged values we can be sure not to incorporate special

[12] This is given with the three month bill rate. The results do not change by taking another variable for the absolute height, for example the 30 year treasury yield, as Duffee (1996a) reports.

effects of one firm. Further it is more realistic to have reliable information about the behavior of one sector than about single firms. This is applied in Barth (1999).

We cannot apply these results directly to our work because we take account of the *relative* changes, while Duffee considers (see eq.(28)) *absolute* changes of the yield spread. This can be seen by regarding the exponential correlation function eq.(23) which could be expanded like

$$\begin{aligned} & S_a(r(t+\Delta t), t+\Delta t) = S_a(r(t), t)\,(1 + k_a \Delta r + \ldots) \qquad (29) \\ \Leftrightarrow \quad & \Delta S_a(r(t), t) = 1 + k_a S_a(r(t), t)\Delta r + \ldots . \end{aligned}$$

Assuming that the difference in the yield spread ΔY only accounts for changes in the market price of default risk (as discussed in Duffee (1996b)), choosing the counterparty a to represent an "average" firm and identifying ΔY and ΔS_a leads to $k_a S_a(r(t), t) = b_1$. By choosing a typical value for S_a it is possible to give a very rough estimate of k_a. We use as typical values the default intensities listed in Fons (1994) for each rating class of Moody's. Table 2 gives the results.

Rating	Aaa	Aa	A	Baa
k_a	-153	-170	-99	-68

Table 2: *Rough estimates of the response coefficient k_a, based on the investigation of Duffee (1996b).*

Even if the absolute responsiveness grows with a lower rating (which is not the case in all regressions of Duffee (1996b) and Düllmann et al. (1998)), this is no longer true with these relative changes. In the numerical investigations in Barth (1999), we use values of k_a for the individual firms in the range of [-32, +32]. The absolute value of these parameters is in all cases much smaller than the values resulting in the study of Duffee. But we consider not only linear but also nonlinear functions like the exponential function, where larger changes of Δr could imply larger contributions. These larger changes will not be important on the monthly scale, but by iterating eq.(30) for many months a larger difference will result. In this sense a careful use of the estimated averaged values is necessary.

3 Conclusion

In this paper a model was presented which is suited for calculating the effects of changes in market variables on the credit risk of a portfolio of market-driven contract subject to credit risk. This model was compared with the frameworks of CreditMetrics and CreditRisk+. In contrast to these frameworks the model proposed here concentrates on the treatment of the market variables while regarding the credit risk. In CreditMetrics and CreditRisk+ the market variables are eliminated and the emphasis is on the treatment of the credit risks. So the model of this paper takes a complementary approach. In the second part of the paper the calibration of the correlation between the market and the default risk is discussed on the basis of recent empirical estimations.

4 Extensions

In Barth (1999) simulation studies based on the model described here are presented. The main subject of these simulation studies is the determination of the risk-based capital. The focus lies on taking properly into account the worst cases of the market variables. Therefore we will discuss several measures for this risk-based capital. The simulation study will refer to these measures.

The points listed below are ideas for extending the work:

- The application of the Law of Large Numbers is unfortunately not valid. In a later version of this paper we want to apply arguments of Nielsen (1985) and Hellwig (1995) to eliminate the individual risks in the case of a large portfolio.
- Instead of considering only one underlying factor for market risk (the interest rate r), the consideration of many underlying factors (for example currency exchange rates) might broaden the view of the model.

References

Albrecht, P., A. König, R. Maurer, and H. R. Schradin (1996): An actuarial approach to determine the required capital for portfolios of options subject to credit risk. Mannheimer Manuskripte Nr.91, Universität Mannheim.

Barth, J. (1999): Credit risk: Worst case scenarios of swap portfolios. Mannheimer Manuskripte Nr. 104, Universität Mannheim.

Berninghaus, S. (1977): Individual and collective risks in large economies. *Journal of Economic Theory*, 15, 279–294.

Cooper, I. A. and A. S. Mello (1991): The default risk of swaps. *Journal of Finance*, 46, 597–620.

CreditSuisse (1997): *CreditRisk+*. Credit Suisse Financial Products.

Daley, D. J. and D. Vere-Jones (1988): *An Introduction to the Theory of Point Processes*. Heidelberg: Springer-Verlag.

Duffee, G. R. (1996a): On measuring credit risks of derivative instruments. *Journal of Banking and Finance*, 20, 805–833.

Duffee, G. R. (1996b): The variation of default risk with treasury yields. In *Risk Measurement and Systemic Risk*, Board of Governors of the Federal Reserve System.

Duffie, D. and M. Huang (1996): Swap rates and credit quality. *Journal of Finance*, 51,3, 921–949.

Duffie, D. and K. J. Singleton (1995): Modeling term structures of defaultable bonds. Working Paper Graduate School of Business, Stanford University.

Duffie, D. and K. J. Singleton (1998): Simulating correlated defaults. Working Paper Stanford University.

Düllmann, K., M. Uhrig-Homburg, and M. Windfuhr (1998): Risk structure of interest rates: An empirical analysis for deutschemark-denominated bonds. Working Paper Universität Mannheim.

Estrella, A. and G. A. Hardouvelis (1991): The term structure as a predictor of real economic activity. *Journal of Finance*, 46,2, 555–576.

Fons, J. S. (1994): Using default rates to model the term structure of credit risk. *Financial Analysts Journal*, 25–32.

Gordy, M. B. (1998): A comparative anatomy of credit risk models. Working Paper Federal Reserve Board.

Grandell, J. (1976): *Doubly Stochastic Poisson Processes*. Heidelberg: Springer-Verlag.

Hellwig, M. F. (1995): The assessment of large compounds of independent gambles. *Journal of Economic Theory*, 67.2, 299–326.

Hull, J. C. (1989): An analysis of the credit risks in interest rate swaps and currency swaps. In *Recent Developments in International Banking and Finance*, 109–130.

J.P.Morgan (1997): *CreditMetrics*. J.P.Morgan.

Longstaff, F. A. and E. S. Schwartz (1995): A simple approach to valuing risky fixed and floating rate dept. *Journal of Finance*, 50, 789–819.

Nielsen, L. T. (1985): Attractive compounds of unattractive investments and gambles. *Scandinavian Journal of Economics*, 87, 463–473.