Risk Pooling and Solvency Regulation: A Policyholder's Perspective*

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1 Introduction

Risk reduction through pooling can be seen as a defining characteristic of the insurance mechanism from the insurer's perspective. For example, Houston (1964, p. 538) argues that:

> "To the individual, insurance is a device for transferring his risk to the insurer. The insurer views insurance as a pooling process in which the risk is reduced by increasing the number of exposure units insured."

Following this view, the benefits of larger risk pools are typically studied by quantifying the reduction of the *insurer's risk* as measured by its default probability or the relative capital buffer (see, e.g., Cummins, 1974, 1991). However, if we take potential losses from a default into account, then the size of the insurance risk pool can also affect the payoffs to policyholders and thus their overall utility from the above-mentioned risk transfer.[1]

In this paper, we reinvestigate the benefits of risk pooling from the policyholders' perspective under different solvency frameworks. We focus on the case of a stock insurer, which is of special interest because the benefits of reducing the risk per policy have to be shared between the policyholders and the owners of the company. This risk allocation can drive a wedge between the occurrence of pooling benefits for the insurer's total position and the policyholders' wealth. Given the limited liability of equity holders, the default risk that the policyholders have to bear for a given amount of total risk depends on the equity capital that the owners of the company provide. We assume that this equity contribution is exogenously determined according to solvency rules and we consider two cases: minimum capital requirements that are proportional to the total premiums written and a capital regulation that is based on the Value-at-Risk (VaR). Capital requirements that are proportional to (net) premiums are an important example for volume-based systems such as the capital charges for underwriting risk in the United States and the former European Solvency I framework. The VaR-based rule has become a main component of probabilistic solvency systems around the world, for example, in the European Solvency II framework.[2]

[1] Several studies have argued that policyholders are highly sensitive to non-performance risks of insurance contracts. Cf. the discussion in Froot (2007) and the literature on "probabilistic insurance" (see, e.g., Wakker et al., 1997; Zimmer et al., 2018) for empirical results. This also applies to markets with guaranty funds whose protection is often only incomplete and associated with additional transaction costs (see Cummins and Sommer, 1996, p. 1075 or Cummins and Weiss, 2016, p. 130).

[2] Holzmüller (2009) and Cummins and Phillips (2009) provide reviews of the solvency regulation in the United States and in Europe. See Geneva Association (2016) for the results of a recent global survey.

Our analysis relies on the following main assumptions: For the risks being insured, we only require homogeneity and finite expectations. Homogeneity is formalized by assuming that the joint distribution of losses is exchangeable, which includes independent and identically distributed losses as a special case. The finiteness of expectations is necessary to evaluate the resulting wealth positions within an expected utility framework. We apply a second degree stochastic dominance criterion to obtain utility comparisons that are consistent with the preferences of risk-averse agents across a wide range of decision models. Default losses for policyholders are modeled endogenously by comparing the total claim amount to the level of the available reserves (equity capital and premiums) following ideas developed by Merton (1974) and Doherty and Garven (1986). Finally, we do not apply a specific pricing model but take the insurance premium as exogenously given. For our baseline analysis, we additionally assume that the policyholders are offered full coverage and that the total default loss from a given pool is shared equally among the policyholders.

These assumptions are sufficient to generate monotonically increasing benefits of risk pooling on the pool level. More specifically, the riskiness of the average claim per policyholder is non-increasing in the pool size under the given assumptions, in line with the general notion that diversification reduces risk. However, we demonstrate that the resulting effect on the policyholders' utility depends on the form of capital regulation due to the asymmetric risk sharing between policyholders and equity holders.

Under a simple volume-based solvency framework in which the equity capital is proportional to the premiums, the pooling benefits for policyholders are consistent with the overall risk reduction. In particular, we show that the policyholders' utility level is non-decreasing in the pool size so that all risk-averse policyholders at least weakly prefer insurance in larger risk pools.

In contrast, the occurrence of a risk reduction on the pool level does not necessarily translate into utility gains for policyholders if the amount of equity capital is determined using the VaR. Although, by construction, a VaR-based equity capital limits the probability of default, the relationship between the pool size and the policyholders' utility level depends on the distribution of the risks that are pooled. Varying the distributional assumption on the losses, we illustrate that the policyholders' utility level can be (i) globally increasing, (ii) locally decreasing or even (iii) globally decreasing in the size of the risk pool. We then derive a condition that is necessary and sufficient for non-negative pooling benefits under a VaR-based regulation. In particular, we show

that all risk-averse policyholders prefer larger risk pools if and only if the excess tail risk of the average claim as measured by the difference between Average Value-at-Risk (AVaR) and VaR does not increase in the size of the pool. In addition, we provide a sufficient condition on the joint distribution of individual risks, which is for example satisfied by risks from a multivariate elliptical distribution with equal correlations and finite variances.

Finally, we investigate a case in which the policyholders also own an equity stake in the insurance company. In this case, the effect of risk pooling on the policyholders' utility is always nonnegative – independent of the form of the minimum capital requirements.

We then discuss several extensions of our baseline analysis: First, we take a variable expense loading into account and confirm the intuition that cost benefits can reinforce risk-related pooling benefits. Moreover, we demonstrate that the benefits of risk pooling are robust to introducing independent investment risk if the equity capital is proportional to the premiums. To obtain a corresponding result under VaR-based capital requirements, we have to impose an additional shape restriction on the distribution of the investment return. We then study the special case of independent risks, which allows us to relax our full coverage and equal loss sharing assumptions and to derive sufficient conditions for utility gains from risk pooling with more general contract types and with alternative sharing rules for the total default loss.[3]

Our analysis is related to the literature on the benefits of risk pooling and diversification. As mentioned above, several authors have studied the relationship between the size of the risk pool and the insurer's risk, often applying asymptotic arguments that build on the law of large numbers or the central limit theorem (see, Houston, 1964; Cummins, 1974, 1991; Smith and Kane, 1994 among others). However, the impact of risk pooling on the policyholders' utility has so far mainly been investigated in the mutual insurance case. In particular, Gatzert and Schmeiser (2012) and Albrecht and Huggenberger (2017) show that policyholders benefit from risk pooling in this setting by exploiting that mutual insurance companies attain a complete sharing of profits and losses independent of premiums or capital reserves. Due to this insight, the analysis of the mutual insurance case is related to general results on diversification benefits for risk-averse decision makers (Samuelson, 1967; Rothschild and Stiglitz, 1971).[4] For the case of stock insurers, we are only aware

[3]In an Online Appendix to this paper, we additionally derive asymptotic results on pooling benefits for policyholders.

[4]Cf. also Eeckhoudt et al. (1993), who study the interaction between diversification and insurance from the

of the recent work by Schmeiser and Orozco-Garcia (2018) who compare pooling benefits in mutual and stock insurance companies focussing on conditions under which policyholders attain the same utility levels from both organizational forms.[5] We extend the previous literature by providing a general analysis of pooling benefits for policyholders in stock insurance companies. This analysis reveals that risk pooling can have a negative impact on the policyholder's utility under a VaR-based regulation.

Given this finding, our work complements a number of recent studies that highlight adverse effects of diversification. Ibragimov (2009) shows that diversification can increase the overall VaR when pooling risks with extremely heavy tails.[6] Ibragimov et al. (2009) document the occurrence of "diversification traps" in reinsurance markets, which are due to locally decreasing utility levels from diversification with, again, heavy tailed risks. Furthermore, Ibragimov et al. (2011) demonstrate that the optimal level of diversification from the perspective of financial intermediaries can have adverse welfare implications.

Our work is also related to the economic literature on the role of default risk for the optimal design and the pricing of insurance contracts. In particular, our analysis of the policyholder's utility under default risk complements the work of Doherty and Schlesinger (1990) and subsequent studies (Cummins and Mahul, 2003; Mahul and Wright, 2004, 2007) that document how well-known results on optimal insurance purchasing have to be modified if indemnity payments are subject to default risk. In addition, our endogenous modeling of default losses is related to the contingent claim approach for pricing default risk (Doherty and Garven, 1986; Cummins, 1988).[7] Despite these similarities, the focus of our analysis is different: We study the impact of risk pooling on the policyholder's utility losses from default – taking insurance prices as well as the offered form of coverage as given.

Finally, our results add to the literature on the optimal design of solvency regulation. While risk-sensitive capital requirements for insurance companies have a number of important advantages

perspective of risk-averse decision makers.

[5]See also Laux and Muermann (2010) or Braun et al. (2015) and the references therein for previous comparisons of mutual insurance and stock insurance companies along different dimensions.

[6]Ibragimov and Walden (2007) obtain similar results for random variables with bounded support that are generated by truncating heavy-tailed distributions. Note that the VaR-based diversification limits documented with this approach do not apply in a utility-based analysis (with concave utility functions).

[7]Furthermore, our methodology for sharing the default loss between policyholders is related to ideas that have been applied to the pricing of default risk in multiline insurance companies (Phillips et al., 1998; Myers and Read, 2001; Ibragimov et al., 2010).

compared to volume-based frameworks (Cummins et al., 1993; Holzmüller, 2009), our analysis highlights the potential problem of the risk-based approach that a reduction of the capital buffer per policy for larger portfolios can adversely affect the policyholder's utility. Moreover, our results contribute to the ongoing debate on the adequacy of VaR for setting capital requirements. On the one hand, VaR has been heavily criticized because it is not always subadditive (Artzner et al., 1999) and does not sufficiently reflect the risk of extreme losses.[8] On the other hand, Dhaene et al. (2008) demonstrate that strong subadditivity can be undesirable for the determination of capital requirements since it can increase the shortfall risk after a merger. Building on this insight, they characterize the subadditivity level of VaR as "efficient".[9] We add a stakeholder-specific perspective to this debate by demonstrating that VaR-based capital requirements can reduce or even eliminate diversification benefits for the policyholders of stock insurers. Interestingly, this effect is not related to a lack of subadditivity in our examples but rather in line with the criticism of subadditivity by Dhaene et al. (2008).

We proceed as follows: Section 2 introduces our main assumptions and our decision-theoretic approach for evaluating pooling benefits. In Section 3, we present our baseline results on the benefits of risk pooling under full coverage and an equal sharing of default losses. Section 4 extends these results by including expenses, investment risk as well as more general forms of coverage and loss sharing schemes. Section 5 concludes. The proofs are given in the Appendix. An Online Appendix presents complementary results and details on our examples.

2 Decision Framework

In this section, we present our baseline assumptions, the solvency standards that we consider and the methodology used for general utility comparisons.

2.1 Baseline Assumptions

We study a one-period model with n agents, $n \in \mathbb{N}$. Agent i possesses the initial wealth $w_{0,i}$ at time $t = 0$ and faces a risk with the potential loss X_i at time $t = 1$, $i = 1, \ldots, n$. To simplify the

[8] Cf., for example, Dowd and Blake (2006) for a comprehensive review of VaR and its alternatives with a focus on insurance applications.

[9] In addition, Kou and Peng (2016) recently show that VaR satisfies an alternative set of axioms and emphasize the robustness of VaR compared to risk measures that are more sensitive to extreme tail events.

exposition, we assume that the risk-free rate is zero.[10] Without insurance, the end-of-period wealth of agent i is then given by

$$W_i = w_{0,i} - X_i. \tag{1}$$

For the distribution of the losses, we mainly rely on the following assumption:

Assumption A1 *The losses* $(X_1, \ldots, X_n)$ *are exchangeable with* $\mathbb{E}[|X_i|] < \infty$ *for all* $i = 1, \ldots, n$.

Exchangeability means that the joint distribution of the random variables $(X_1, \ldots, X_n)$ does not change under permutations of the indices (McNeil et al., 2015, p. 234).[11] This implies that the marginal distributions of the losses are identical. Assumption A1 thus captures the notion of a homogeneous risk pool in a rather general way. In particular, it includes the important special case of identically distributed and independent risks (IID). Compared to this special case, exchangeability allows for positive dependence. With respect to the marginal distributions, Assumption A1 does only require that the risks are absolutely integrable, but it does not restrict our analysis to specific parametric models.[12]

In our baseline analysis, we focus on insurance policies that offer full coverage:[13]

Assumption A2 *Agent* i *can buy full coverage of her loss* X_i *for the risk premium* $\pi > 0$, $i = 1, \ldots, n$.

We take the risk premium π as given instead of imposing a specific pricing rule and assume that every agent is offered the contract at the same price, i.e. $\pi_i = \pi$, which is a natural assumption for identically distributed risks. The premium π can be understood as the prevailing market price for full coverage, which is exogenous and independent of the insurance company selling the contract.[14] In particular, Assumption A2 implies that the premium does not vary with the number of policies sold.[15]

If agent i is able to buy an insurance policy that is not subject to default risk, her end-of-period

[10] This assumption can easily be relaxed and does not affect our main conclusions.

[11] Cf., e.g., Denuit and Vermandele (1998) as well as Albrecht and Huggenberger (2017) and the references therein for actuarial applications of exchangeable random variables.

[12] Usually, it holds that $X_i \geq 0$. However, we do not need this additional restriction for our general analysis.

[13] Alternative types of contracts will be studied in Section 4.3.

[14] Empirical evidence for a limited degree of premium differentiation even across organizational forms of insurance companies is e.g. provided by Braun et al. (2015).

[15] Variable premium loadings that depend on the size of the risk pool will be analyzed in Section 4.1.

wealth at $t = 1$ satisfies[16]

$$W_i^s = w_{0,i} - \pi. \tag{2}$$

This wealth position does not depend on the size of the risk pool, which is a crucial difference to the mutual insurance case, where the size of the pool directly affects the distribution of the policyholders' wealth as a result of their profit participation (Gatzert and Schmeiser, 2012; Albrecht and Huggenberger, 2017).

However, the simple position in equation (2) neglects that the funds of the insurance company are usually limited and that it might not be able to cover all claims at $t = 1$. We refer to insurance policies that are subject to this kind of default risk as *vulnerable contracts* (Johnson and Stulz, 1987; Cummins and Mahul, 2003) and we let $D_{i,n}$ denote the default loss of policyholder i, who bought insurance from a company with a total portfolio of n policies.[17] The final wealth of policyholder i from buying a vulnerable contract is then given by

$$W_{i,n} = w_{0,i} - \pi - D_{i,n}. \tag{3}$$

Our approach for modeling the distribution of $D_{i,n}$ relies on the assumption that all contracts are offered by a stock insurance company with limited liability. If the company sells n policies, its total claim amount can be calculated as

$$S_n = \sum_{i=1}^{n} X_i. \tag{4}$$

The funds available to cover these claims are the premium payments made by the policyholders and the equity capital that the owners of the insurance company provide at $t = 0$. The premiums correspond to $n\,\pi$ and the total equity available for a portfolio of size n is denoted by c_n. Due to the limited liability of the owners, default occurs if and only if

$$S_n > c_n + \pi\, n. \tag{5}$$

[16] This analysis ignores any additional insurable or non-insurable risks of the decision maker, particularly background risk (Schlesinger, 2013, pp. 180ff.) as well as any additional income (labour income, additional investment income, and/or pension benefits).

[17] Here, n is the total size of the risk pool including the contract of policyholder i.

In this setting, the default probability for a pool of size n satisfies

$$PD_n = \mathbb{P}[S_n > c_n + n\,\pi] \tag{6}$$

and the total default loss is given by

$$L_n = \max(S_n - c_n - n\,\pi,\, 0). \tag{7}$$

L_n is the amount missing for fully covering all claims from the contracts in the pool. We thus work with default losses that are *endogenously* caused by high claim amounts.[18] $-L_n = -\max(S_n - c_n - n\,\pi,\, 0)$ corresponds to the payoff of the well-known default option which the policyholders implicitly sell to the owners of the company.[19]

For most of our analysis, we assume that policyholders and owners are distinct groups.[20] To fully describe the impact of default on the *individual* policyholder's wealth, it then remains to define a rule for sharing the total default loss L_n from equation (7) among the policyholders.[21] A natural first choice is an equal distribution of L_n.[22] In this case, every policyholder's wealth is reduced by

$$\bar{L}_n := \frac{1}{n} L_n. \tag{8}$$

In our setting with identically distributed losses, this corresponds to an "ex ante" sharing rule, which splits the total default loss according to the policyholders' contribution to the total expected loss.[23] If we denote the average claim per policyholder by $\bar{S}_n := S_n/n$ and the available equity capital per policyholder by $\bar{c}_n := c_n/n$, then equation (8) can be rewritten as $\bar{L}_n = \max(\bar{S}_n - \bar{c}_n - \pi,\, 0)$.

The formal implications of the previous discussion are summarized in the following assumption:

Assumption A3 *The default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = \bar{L}_n$,*

[18] In Section 4.2, we introduce investment risk as an additional source of default losses.

[19] The idea of using contingent claim pricing to analyze corporate debt goes back to Merton (1974). In an insurance context, it has been introduced by Doherty and Garven (1986) and Cummins (1988).

[20] A combined policyholder and owner position will be investigated in Section 3.4.

[21] By dividing the *total* excess loss among the policyholders, we implicitly assume that the policyholders' claims are not protected by a guaranty fund. In Section 4.3, we will introduce a more general modeling of individual default losses that allows for partial compensation by a guaranty fund.

[22] See, for example, Gatzert and Schmeiser (2012, p. 191) for the equal sharing of the excess loss.

[23] Cf. Ibragimov et al. (2010) for a discussion of "ex ante" vs. "ex post" sharing rules. We will consider more general sharing rules in Section 4.3.

$i = 1, \ldots, n$.

Under the Assumptions A2 and A3, we can rewrite the policyholder's wealth from buying a vulnerable insurance contract as

$$W_{i,n} = w_{0,i} - \pi - \bar{L}_n = w_{0,i} - \pi - \max(\bar{S}_n - \bar{c}_n - \pi,\, 0). \tag{9}$$

Equations (6) and (9) show that the negative effect of an increase in the premium is partially compensated by a reduction of the default probability or the magnitude of the default loss. Furthermore, equation (9) reveals that the size of the risk pool affects the policyholder's wealth under the given assumptions through two channels: the distribution of $\bar{S}_n$ and the available equity capital per policyholder $\bar{c}_n$. The dependence of the policyholder's wealth on the size of the risk pool is an important difference to the case without default risk as it potentially generates benefits of risk pooling from the policyholders' perspective beyond the mutual insurance case.

2.2 Capital Regulation

To understand the effect of different solvency rules, we assume that the equity capital provided by the owners exactly corresponds to the minimum capital requirement for a given pool size. In particular, we investigate the benefits of risk pooling under the following two solvency rules.

First, we consider the case in which the risk capital c_n is proportional to the total premiums written $n\,\pi$, which captures volume-based solvency frameworks like the underwriting risk charges according to the RBC standards in the United States or the former European Solvency I rules (Cummins and Phillips, 2009; Holzmüller, 2009). Since we assume a homogeneous pool with an identical risk premium for all contracts, this type of regulation implies that the minimum equity capital increases proportionally with the size of the risk pool n, i.e.,

$$c_n = n\,c. \tag{10}$$

Equity holders then provide the same amount of equity capital $\bar{c}_n = c$ for each contract.

Second, we consider minimum capital requirements that are based on the Value-at-Risk (VaR). VaR-based capital requirements are a very important component of modern probabilistic solvency

standards in many insurance markets around the world (Geneva Association, 2016, Table 1), such as the current European Solvency II framework. We define the VaR of the random loss L at the probability level α as

$$\mathrm{VaR}_\alpha[L] = Q_{1-\alpha}[L] \tag{11}$$

with $Q_u[X]$ denoting the u-quantile of the random variable X, i.e., $Q_u[X] = \inf\{x \in \mathbb{R};\, \mathbb{P}[X > x] \leq 1 - u\}$. Under a VaR_α-based capital regulation, the minimum equity capital is calculated as

$$c_n = \mathrm{VaR}_\alpha[S_n - \pi\, n]\,. \tag{12}$$

By the definition of the VaR_α, this choice of the equity capital ensures that the insurer's default probability does not exceed the probability level α. Note that equation (12) can be rewritten as

$$\bar{c}_n = \mathrm{VaR}_\alpha\left[\bar{S}_n - \pi\right], \tag{13}$$

which typically implies that the equity capital per policyholder varies with the size of the risk pool. We illustrate this dependence for the simple case of normally distributed risks in the following example.

Example 1 *Suppose that the losses $(X_1, \ldots, X_n)$ are independent and normally distributed with $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i = 1, \ldots, n$, where $\sigma > 0$. Under Assumption A2, we obtain $S_n \sim \mathcal{N}(n\mu, n\sigma^2)$ for the total claim amount and $\bar{S}_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\sigma^2\right)$ for the average claim amount. Therefore, the available equity capital per policyholder according to equation* (13) *is given by*

$$\bar{c}_n = Q_{1-\alpha}[\bar{S}_n] - \pi = \mu - \pi + \Phi^{-1}(1-\alpha)\,\frac{1}{\sqrt{n}}\sigma \tag{14}$$

with $\Phi^{-1}(1-\alpha)$ denoting the $(1-\alpha)$-quantile of the standard normal distribution. For $\alpha < 0.5$, it holds that $\Phi^{-1}(1-\alpha) > 0$ and the equity capital per policyholder is decreasing in the pool size n.

The negative relationship between the amount of risk capital per policyholder that is required to maintain a given safety level as measured by the default probability is sometimes interpreted as a "benefit of risk pooling".[24] However, this classification only applies to the equity holders'

[24] See e.g. Gatzert and Schmeiser (2012), who refer to this effect as "case A" risk pooling.

perspective, who have to provide less capital per policyholder, unless this advantage is redistributed to policyholders through a lower loading on the risk premium.

2.3 Stochastic Orders and Preferences

To study the impact of the pool size n on $W_{i,n}$ according to equation (3), we rely on the theory of decisions under risk. Choosing this framework, we implicitly assume that the policyholders cannot replicate their wealth positions using tradable assets. If a complete replication of the corresponding positions was possible, their value would correspond to the price of the respective replicating portfolio. However, it seems reasonable that a typical policyholder does not have access to instruments which can be used to replicate cashflows depending on the individual losses $X_1, \ldots, X_n$.

More specifically, we use second degree stochastic dominance (SSD) for utility comparisons that are largely independent of a particular preference specification. Let W_1 and W_2 denote random wealth positions. W_1 is said to dominate W_2 by SSD ($W_1 \succeq_{ssd} W_2$) if $\mathbb{E}[\psi(W_1)] \geq \mathbb{E}[\psi(W_2)]$ for all non-decreasing concave functions ψ such that the expectations exist.[25] In the standard expected utility theory (EUT), this definition directly implies that all risk-averse agents with a utility function u_i satisfying $u_i' > 0$ and $u_i'' < 0$ weakly prefer W_1 over W_2 if $W_1 \succeq_{ssd} W_2$.

Given the limited ability of EUT to explain the findings of the empirical literature on "probabilistic insurance" (Wakker et al., 1997), it is important to note that our SSD analysis is also consistent with decision theories that incorporate probability weighting. In particular, SSD results reflect the preferences of risk-averse agents in the rank-dependent expected utility model (Quiggin, 1982; Yaari, 1987) if the utility function and the probability weighting function are concave. Formally, we assume that agent i assigns the value

$$V_i(W) = \int u_i(w)\, d(g_i \circ F_W)(w) \tag{15}$$

to W, where u_i denotes the agent's utility function with $u_i' > 0$ and $u_i'' < 0$, F_W is the cumulative distribution function of W and g_i is an agent-specific probability weighting function with $g_i(0) = 0$,

[25] Cf. Hadar and Russell (1969) as well as Rothschild and Stiglitz (1970). Our analysis relies on the comprehensive discussion in Shaked and Shanthikumar (2007, Chapter 4), where SSD is introduced as increasing concave order. See also Levy (1992) for an extensive review of economic applications.

$g_i(1) = 1$, $g_i' > 0$ and $g_i'' < 0$.[26] Then, it follows from Chew et al. (1987) that

$$W_1 \succeq_{ssd} W_2 \qquad \Rightarrow \qquad V_i(W_1) \geq V_i(W_2)\,. \tag{16}$$

Further relevant properties of SSD that will be used for analyzing the policyholders' utility are summarized in the Appendix.

3 Baseline Results

In this section, we investigate the benefits of risk pooling under the baseline assumptions introduced in the previous section. First, we briefly describe the impact of risk pooling on the average risk of an insurance portfolio. Then, we present our main results on the policyholders' utility for a proportional growth of the available equity capital in Section 3.2 and for a Value-at-Risk-based rule in Section 3.3. In these analyses, we assume that policyholders and equity holders are distinct groups. In Section 3.4, we finally study the potential benefits of larger risk pools if the policyholders also provide the equity capital.

3.1 The Average Risk of the Pool

Under our baseline assumptions, the intuition that pooling reduces the risk per policyholder can be formalized as follows.

Lemma 1 *Suppose that the losses* $(X_1, \ldots, X_{n+1})$*,* $n \in \mathbb{N}$*, satisfy Assumption A1 and let* $\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ *denote the average loss of* $(X_1, \ldots, X_n)$*, then*

$$-\bar{S}_{n+1} \succeq_{ssd} -\bar{S}_n \qquad \textit{for all } n \geq 1. \tag{17}$$

Proof: The result is obtained by applying Lemma 3.1 from Albrecht and Huggenberger (2017) to $(-X_1, \ldots, -X_{n+1})$.

[26] Note that this value functional is related to the more general Choquet Expected Utility framework following Schmeidler (1989). In this framework, our assumption of a concave probability weighting function corresponds to a convex capacity. A representation of equations (15) and (16) in terms of Choquet Expected Utility can be found in Albrecht and Huggenberger (2017).

Equation (17) states that all risk-averse decision makers prefer the average loss from a larger pool to the average loss from a smaller pool. Similar results, in particular for independent and identically distributed risks, are frequently exploited in the literature on risk pooling and diversification and can (at least) be traced back to Samuelson (1967) and Rothschild and Stiglitz (1971).

To simplify the following presentation, it is helpful to rewrite the result of Lemma 1 in terms of increasing convex order (Shaked and Shanthikumar, 2007, Chapter 4), which is often referred to as stop-loss order in the actuarial literature (Denuit et al., 2005, p. 152). X is said to be smaller than Y in increasing convex order ($X \preceq_{icx} Y$) if $\mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)]$ for all non-decreasing convex functions ψ such that the expectations exist. With this definition, equation (17) is equivalent to[27]

$$\bar{S}_{n+1} \preceq_{icx} \bar{S}_n \qquad \text{for all } n \geq 1. \tag{18}$$

Accordingly the *average* risk of the pool is decreasing in the pool size if "risk" is measured in terms of increasing convex order.[28]

In the remaining part of this section, we investigate whether the policyholders whose contracts are pooled can benefit from this risk reduction on the pool level.

3.2 Volume-Based Capital Requirements

We first focus on a volume-based regulation with a constant amount of equity capital per policy. In this case, the default loss per policyholder is given by

$$\bar{L}_n^c = \max(\bar{S}_n - c - \pi,\, 0) \tag{19}$$

and we are able to derive the following result.

Proposition 1 *Suppose that the Assumptions A1, A2 and A3 hold. If the equity capital per policyholder is constant $\bar{c}_n = c$, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W_{i,n+1} \succeq_{ssd} W_{i,n}$ for all $n \geq 1$.*

Proof: See the Appendix.

[27] The equivalence follows from property i) of Lemma 2 in the Appendix.

[28] We use "decreasing" for "non-increasing" and "increasing" for "non-decreasing" throughout this article.

Proposition 1 states that risk pooling with a larger number of homogeneous risks is always more beneficial from the policyholders' perspective. As a consequence[29] of equation (16), all risk-averse agents with preferences according to equation (15) weakly prefer to buy insurance from a company with a larger risk pool, i.e.,[30]

$$V_i(W_{i,n+1}) \geq V_i(W_{i,n}) \qquad \text{for all } n \geq 1. \tag{20}$$

The benefits for policyholders are thus consistent with the results stated for the average risk of the pool at the beginning of this section.

The utility gains from insurance with larger risk pools originate from a decrease in the disutility from default states. In particular, the proof of Proposition 1 shows that

$$V_i\Big(-\bar{L}^c_{n+1}\Big) \geq V_i\Big(-\bar{L}^c_n\Big) \qquad \text{for all } n \geq 1. \tag{21}$$

In the Online Appendix, we complement these results for finite n by an asymptotic analysis within the standard expected utility framework assuming independent and bounded risks. Our results for this case show that the utility losses from default asymptotically vanish as $n \to \infty$ if the reserves per policyholder $(\pi + c)$ cover the expected claim amount (see Proposition I.1).

We now illustrate the result of Proposition 1 for normally distributed risks and agents with exponential utility.[31]

Example 2 *Suppose that the risk preferences of the agents are given by* $u_i(w) = 1 - \exp(-\gamma_i \cdot w)$ *with the risk aversion parameter* $\gamma_i > 0$ *and* $g_i(p) = p$. *According to equation* (9), *the expected utility from buying a vulnerable contract can be written as*

$$\mathbb{E}[u_i(W_{i,n})] = \mathbb{E}\Big[1 - \exp\Big(-\gamma_i\,(w_{0,i} - \pi - \bar{L}^c_n)\Big)\Big] = 1 - \exp\left(-\gamma_i\,(w_{0,i} - \pi)\right)\,\mathbb{M}_{\bar{L}^c_n}(\gamma_i), \tag{22}$$

where $\mathbb{M}_{\bar{L}^c_n}$ *is the moment-generating function of* $\bar{L}^c_n$.

Furthermore, we again consider normally distributed and independent risks $X_i \sim \mathcal{N}(\mu, \sigma^2)$,

[29] This extension from SSD statements to preference relations according to the rank-dependent expected utility model based on equation (16) equally applies to our following results, but we will confine the remaining discussion to SSD statements for the sake of brevity.

[30] Note that establishing a *strict* SSD relation is not possible as explained in the proof of Proposition 1.

[31] More detailed derivations for the presented examples can be found in the Online Appendix.

$i = 1, \ldots, n$. For this case, we know from Example 1 that $\bar{S}_n \sim \mathcal{N}(\mu, \frac{1}{n}\sigma^2)$ and we can exploit results on the partial moment-generating function of the normal distribution presented in the Online Appendix to show that

$$\mathbb{M}_{\bar{L}_n^c}(\gamma_i) = \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \left[1 - \Phi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] + \Phi\left(\frac{\sqrt{n}\, l}{\sigma}\right) \tag{23}$$

with $l = \pi + c - \mu$. Based on equations (22) and (23), it follows that $\mathbb{M}_{\bar{L}_n^c}(\gamma_i)$ is decreasing in n, so that $\mathbb{E}[u_i(W_{i,n})]$ is increasing in the pool size, in line with the general result in equation (20).

In Figure 1, we illustrate the relationship between n and the certainty equivalent of $W_{i,n}$. We assume $\gamma_i = 0.5$, $\mu = 2$ and $\sigma = 4$. For the initial wealth $w_{0,i}$, the premium π and the available equity capital per policyholder c, we use $w_{0,i} = 10$, $\pi = 2$ and $c = 1$. Panel A of Figure 1 illustrates the monotonic decrease of the default probability in the pool size and Panel B shows the monotonic increase of the certainty equivalent. Furthermore, we include the certainty equivalent of buying a safe insurance policy and document that the certainty equivalent of $W_{i,n}$ converges to the level of the safe contract.[32]

3.3 VaR-Based Capital Requirements

We now turn to the VaR-based rule for the determination of the equity capital according to equation (12). The resulting default loss per policyholder is then given by

$$\bar{L}_n^v = \max(\bar{S}_n - \mathrm{VaR}_\alpha\left[\bar{S}_n - \pi\right] - \pi,\, 0) = \max(\bar{S}_n - Q_{1-\alpha}[\bar{S}_n],\, 0), \tag{24}$$

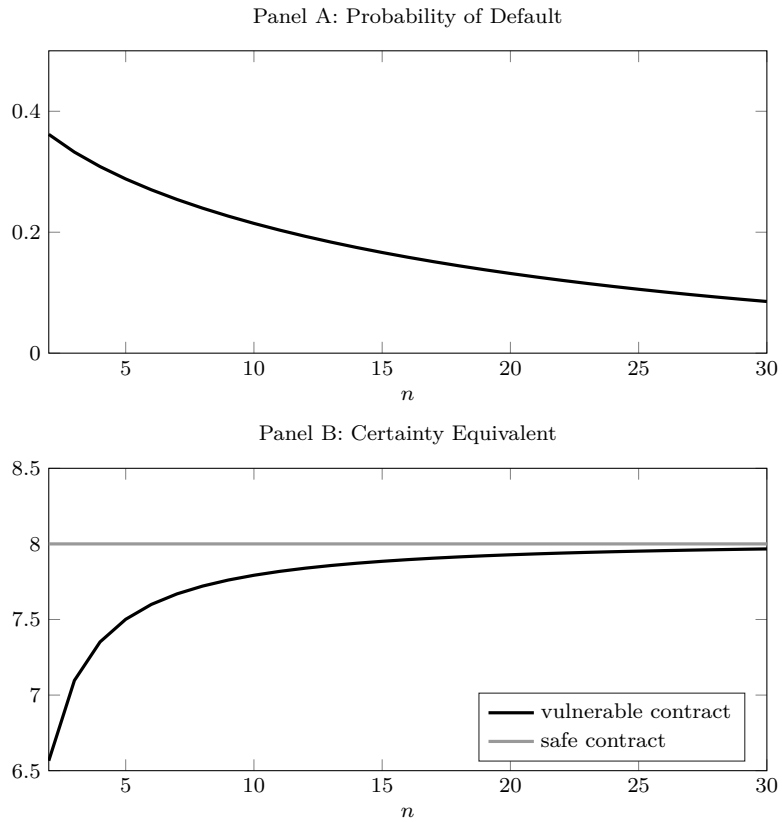
where we exploit equation (13) and $Q_{1-\alpha}[\bar{S}_n - \pi] + \pi = Q_{1-\alpha}[\bar{S}_n]$.

As already pointed out, this situation is highly relevant from a practical perspective given the current capital regulation in several insurance markets. Moreover, it is also of special interest from an economic point of view because a VaR-based capital rule implies in many cases that the equity per policyholder is *decreasing* in the size of the risk pool as illustrated in Example 1.

From the policyholders' perspective, a decreasing capital level can potentially off-set the di-

[32] See Example I.1 in the Online Appendix for formal results on the limit behavior of the utility losses from default under the given assumptions.

Figure 1: Volume-Based Capital Requirements



Note: This figure illustrates a case in which the total equity capital grows proportionally with the premiums and thus with the size of the pool. It builds on the distributional assumptions and preferences presented in Example 2. We assume that $X_i \sim \mathcal{N}(2, 4^2)$, $w_{0,i} = 10$, $\pi = 2$, $c = 1$ and use an exponential utility function with $\gamma_i = 0.5$. Panel A shows how the resulting default probability varies with the pool size n. Panel B depicts the certainty equivalents of buying a safe and a vulnerable insurance contract as a function of n.

versification benefits on the pool level stated in Lemma 1.[33] If this is the case, only the equity holders but not the policyholders benefit from risk pooling. Interestingly, this adverse effect for policyholders is not caused by the lack of subadditivity, which the VaR is often criticized for, but it is a potential problem in cases *satisfying* the subadditivity condition. From this perspective, the given problem is related to Dhaene et al. (2008), who analyze whether risk measures can be "*too* subadditive" in the context of setting capital requirements.

The following example illustrates the different relationships between the pool size and the policyholder's utility that can arise under a VaR-based capital regulation.[34]

Example 3 *We maintain the preference specification introduced in Example 2. Analogously to equation* (22)*, the expected utility of the vulnerable contract under a VaR-based capital rule is given by*

$$\mathbb{E}[u_i(W_{i,n})] = 1 - \exp\left(-\gamma_i(w_{0,i} - \pi)\right) \mathbb{M}_{\bar{L}_n^v}(\gamma_i) \tag{25}$$

with $\mathbb{M}_{\bar{L}_n^v}(\gamma_i)$ *denoting the moment-generating function of* $\bar{L}_n^v$ *defined in equation* (24)*. As in Example 2, we again choose* $\pi = 2$*,* $w_{0,i} = 10$ *and* $\gamma_i = 0.5$ *for our numerical illustrations. Furthermore, we use* $\alpha = 0.05$ *as probability level to determine the VaR-based equity capital.*

We compare three distributional assumptions for the losses $(X_1, \ldots, X_n)$*:*

i) First, we again analyze normally distributed and independent losses $X_i \sim \mathcal{N}(\mu, \sigma^2)$*. For this case, we can use* $\mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{1}{\sqrt{n}}\,\sigma\,\Phi^{-1}(1-\alpha)$ *to derive the general result*

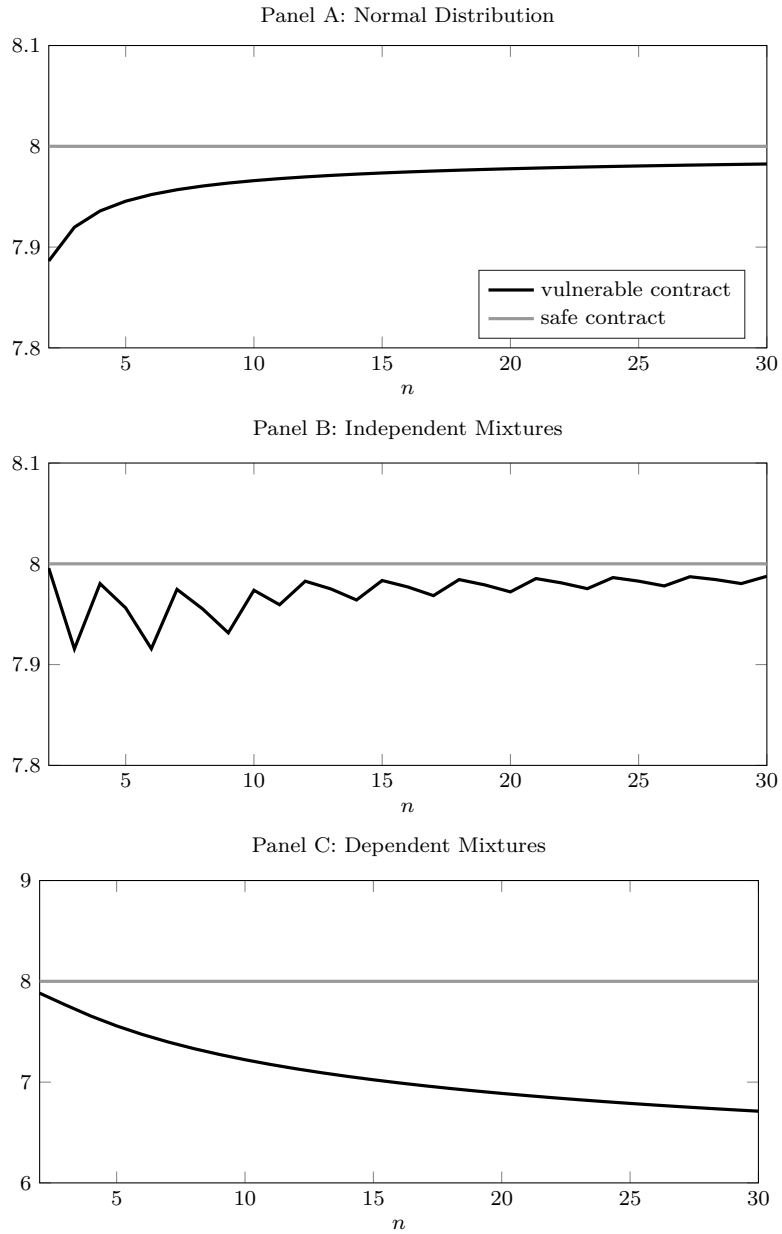
$$\begin{aligned}\mathbb{M}_{\bar{L}_n^v}(\gamma_i) = \exp&\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha) + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\\ &\cdot\left[1 - \Phi\left(\Phi^{-1}(1-\alpha) - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] + (1-\alpha).\end{aligned} \tag{26}$$

Building on this expression, it can be shown that the policyholders' expected utility according to equation (25) *is increasing in* n*. In Panel A of Figure 2, we graphically illustrate this increasing relationship using the same distribution parameters as in Example 2.*[35]

[33] From the potential policyholder's perspective, there is consequently a *trade-off* between diversification benefits in larger risk pools and lower levels of risk capital provided by the equity holders.

[34] Detailed calculations for this example as well as an additional example with a discrete loss distribution can be

Figure 2: VaR-Based Capital Requirements – Certainty Equivalents



Note: This figure illustrates the case of a VaR-based equity capital under the distributional assumptions and preferences used in Example 3. It depicts the certainty equivalents of buying the safe insurance contract (gray line) and the vulnerable insurance contract (black line) as a function of the pool size n. For all examples, we use an exponential utility function with $\gamma_i = 0.5$ and suppose that $w_{0,i} = 10$, $\pi = 2$ and $\alpha = 0.05$. The functions in Panel A are derived from independent and identically distributed losses with $X_i \sim \mathcal{N}(2, 4^2)$. For the illustration in Panel B, we assume that the losses are independent and follow a two-state normal mixture, whose cdf is given by equation (27) with $p_L = 3/4$, $p_H = 1/4$, $\mu_L = 0$, $\mu_H = 8$ and $\sigma_L = \sigma_H = 0.1$. The graph in Panel C also relies on a normal mixture with two states for X_i, but assumes a common state indicator as detailed in part iii) of Example 3. The mixture parameters for the illustration in Panel C correspond to $p_L = 0.96$, $p_H = 0.04$, $\mu_L = 2$, $\mu_H = 4$, $\sigma_L = 4$ as well as $\sigma_H = 1$.

ii) Second, we assume that the distribution of X_i is a normal mixture with two components given by

$$\mathbb{P}[X_i \leq x] = \mathbb{P}[Y_i = L]\ \Phi(x; \mu_L, \sigma_L^2) + \mathbb{P}[Y_i = H]\ \Phi(x; \mu_H, \sigma_H^2), \tag{27}$$

where μ_L (μ_H) and σ_L^2 (σ_H^2) denote the location and variance parameters in a low (high) loss state and $\Phi(\cdot; \mu, \sigma^2)$ denotes the cumulative distribution function of a normal distribution with mean μ and variance σ^2. Y_i is the state indicator for X_i with $\mathbb{P}[Y_i = L] = p_L > 0$, $\mathbb{P}[Y_i = H] = p_H > 0$ for all $i = 1, \ldots, n$ and $p_L + p_H = 1$. We assume that the state indicators $(Y_1, \ldots, Y_n)$ are independent. Then, the losses $(X_1, \ldots, X_n)$ are also independent and $\bar{S}_n$ follows a normal mixture distribution with $n + 1$ components.

For this case, we derive an analytical representation of $\mathbb{M}_{\bar{L}_n^v}(\gamma_i)$ in the Online Appendix. It turns out that the the sign of the relationship between the pool size and the policyholder's expected utility depends on the range of n and the distribution parameters. The occurrence of a non-monotonic relationship is illustrated in Panel B of Figure 2 for $\mu_L = 0$, $\mu_H = 8$, $\sigma_L = \sigma_H = 0.1$, $\pi_L = \frac{3}{4}$ and $p_H = \frac{1}{4}$.[36] Although the certainty equivalent of the vulnerable contract approaches the level of the safe policy for large n, it can be preferable for risk-averse policyholders to participate in a smaller rather than in a larger risk pool for some values of n.

iii) Third, we maintain the mixture assumption from equation (27) for the individual losses but we now assume that the state indicators are perfectly dependent, i.e, $\mathbb{P}[Y_i = Y_j] = 1$ for all $i, j = 1, \ldots, n$. The random variables $Y_1, \ldots, Y_n$ can thus be replaced by a single state variable, which is referred to as common mixture modeling in the actuarial literature (Wang, 1998). $\bar{S}_n$ then follows a two-state mixture and our results in the Online Appendix show that the expected utility of buying the vulnerable contract can even be monotonically decreasing *in the size of the risk pool. This is illustrated in Panel C of Figure 2 for $\pi_L = 0.96$, $\pi_H = 0.04$, $\mu_L = 2$, $\mu_H = 10$, $\sigma_L = 4$ and $\sigma_H = 1$. These parameters imply an unconditional Pearson correlation between the risks of only 0.14 but the certainty equivalent of buying the insurance contract*

found in the Online Appendix.

[35] The graph again indicates the convergence to the utility level of the safe insurance contract. See Example I.2 in the Online Appendix for a formal asymptotic analysis of the policyholders' position.

[36] These assumptions imply $\mathbb{E}[X_i] = 2$ and $\sigma[X_i] = 3.47$, so that the first two moments are roughly comparable to the normally distributed losses that we analyzed before. However, the mixture distribution is asymmetric with a skewness coefficient of 1.15.

subject to default risk turns out to be decreasing in n.

The parts ii) and iii) of the previous example clearly show that risk pooling with a larger number of policies can reduce the expected utility under a VaR-based capital rule – even if Assumption A1 is satisfied. In these cases, the policyholders' position can be adversely affected by an increase in the pool size even though $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ holds for all $n \geq 1$. We thus need additional restrictions to identify cases, in which larger risk pools are beneficial for policyholders with VaR-based minimum capital requirements.

For this purpose, we next present a necessary *and* sufficient condition, which relies on the so-called excess wealth order. According to Shaked and Shanthikumar (2007, p. 164), X is said to be smaller than Y in excess wealth order ($X \preceq_{ew} Y$) if

$$\int_{Q_u[X]}^{\infty} (1 - F_X(x))dx \leq \int_{Q_u[Y]}^{\infty} (1 - F_Y(y))dy \qquad \text{for all } u \in (0,1). \tag{28}$$

In an actuarial context, excess wealth order has for example been studied by Denuit and Vermandele (1999) or Sordo (2008, 2009).

Using this stochastic order, the assumption that we need to establish pooling benefits under VaR-based capital requirements can be written as follows:

Assumption A4 $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ *for all* $n \geq 1$.

Given $\mathbb{E}\left[\bar{S}_n\right] = \mathbb{E}\left[\bar{S}_{n+1}\right]$, $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ implies $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ but the converse does not necessarily hold (Shaked and Shanthikumar, 2007, p. 166 and Theorem 4.A.34). In this sense, Assumption A4 is *more* restrictive for the average risk of the pool than Assumption A1, which only implies $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$.

To understand the economic content of Assumption A4, we characterize excess wealth order in terms of VaR and Average Value-at-Risk (AVaR). The AVaR of a random loss X at the probability level α is given by[37]

$$\text{AVaR}_\alpha[X] = \frac{1}{\alpha} \int_{1-\alpha}^{1} Q_u[X]du. \tag{29}$$

[37] We follow the terminology of Föllmer and Schied (2016). The AVaR is also known as Tail Value at Risk (Dhaene et al., 2006) or Expected Shortfall (Acerbi and Tasche, 2002).

Building on this definition, we can rewrite equation (28) as follows[38]

$$\mathrm{AVaR}_\alpha[X] - \mathrm{VaR}_\alpha[X] \leq \mathrm{AVaR}_\alpha[Y] - \mathrm{VaR}_\alpha[Y] \qquad \text{for all } \alpha \in (0,1). \tag{30}$$

Assumption A4 can thus be interpreted as a decrease in the "excess tail risk" beyond the VaR on the level of the average claim per policyholder.

Using this condition, we are able to establish the following result on the benefits of risk pooling under a VaR-based capital regulation.

Proposition 2 *Suppose that the Assumptions A2 and A3 hold. Furthermore, assume that the equity capital is given by* $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$. *Then, the utility benefits of risk pooling under default risk are increasing in the pool size for all* $\alpha \in (0,1)$ *and* $n \geq 1$, *i.e.,* $W_{i,n+1} \succeq_{ssd} W_{i,n}$, *if and only if Assumption A4 is satisfied.*

Proof: See the Appendix.

According to Proposition 2, every risk-averse agent prefers pooling her risk with a larger number of policies if the excess wealth order requirement for the average claim per policyholder is satisfied.[39] Furthermore, this requirement turns out to be a necessary condition for establishing utility gains under a VaR-based regulation.

To illustrate this result, we verify that the condition in Assumption A4 can be used to distinguish between the cases presented in Example 3.

Example 4 *Under the normality assumption used in part i) of Example 3, it is straightforward to show that Assumption A4 holds. From* $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{1}{\sqrt{n}}\,\sigma\,\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]]$ *with* $Z \sim \mathcal{N}(0,1)$ *and equation* (30)*, it follows that Assumption A4 is equivalent to*

$$\frac{1}{\sqrt{n+1}}\,\sigma\left(\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] - Q_{1-\alpha}[Z]\right) \leq \frac{1}{\sqrt{n}}\,\sigma\left(\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] - Q_{1-\alpha}[Z]\right). \tag{31}$$

This inequality is satisfied for all n *due to* $\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] \geq Q_{1-\alpha}[Z]$. *Panel A of Figure 3 illustrates the corresponding decrease in excess tail risk for the parameters used in Example 3 i).*

[38] Cf. Shaked and Shanthikumar (2007, Eq. 3.C.3) or Sordo (2008, Definition 3 and Eq. 6) together with Dhaene et al. (2006, Theorem 2.1) for this characterization.

[39] An alternative *sufficient* condition for utility gains from risk pooling in the case of a VaR-based regulation is that $\bar{S}_{n+1}$ is smaller than $\bar{S}_n$ in dispersive order. See, e.g., Shaked and Shanthikumar (2007, p. 166) for the relationship between dispersive order and excess wealth order.

In line with Proposition 2, Panel B and Panel C of Figure 3 show that the relationship between the pool size and the excess tail risk as measured by $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ *is* not *decreasing under the distributional assumptions used in the parts ii) and iii) of Example 3.*

Motivated by the first part of this example, we introduce the following more elementary restriction on the joint distribution of the individual claims.

Assumption A5 $\sum_{i=1}^n X_i \stackrel{d}{=} a_n + b_n Z$, $a_n, b_n \in \mathbb{R}$ *with* $\mathbb{E}[Z^2] < \infty$ *for all* $n \in \mathbb{N}$.

Assumption A5 requires that the distribution of the total claim amount $S_n = \sum_{i=1}^n X_i$ is in the same location-scale family for all n and that its variance is finite.[40] Adding this property to our original Assumption A1, we can state the following modification of Proposition 2:

Corollary 1 *Suppose that the Assumptions A1, A2, A3 and A5 hold. If the equity capital is given by* $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$*, then the utility benefits of risk pooling are increasing in the pool size, i.e.,* $W_{i,n+1} \succeq_{ssd} W_{i,n}$ *for all* $n \geq 1$.

Proof: See the Appendix.

Note that the combination of Assumption A1 and A5 is only sufficient but not necessary in contrast to the excess wealth order condition (Assumption A4). An important class of distributions satisfying Assumption A5 are multivariate elliptical distributions (Owen and Rabinovitch, 1983; Landsman and Valdez, 2003; Hamada and Valdez, 2008).

The additional asymptotic results that we provide in the Online Appendix also apply to VaR-based capital standards (see again Proposition I.1). As for the case of a constant equity per policyholder, we basically find that the utility losses go to zero as $n \to \infty$ if the reserves are sufficiently large to cover the expected claim amount. Accordingly, the results based on standard asymptotic diversification arguments do not reflect the different consequences of risk pooling for the policyholders' utility documented in our analysis for finite n.[41]

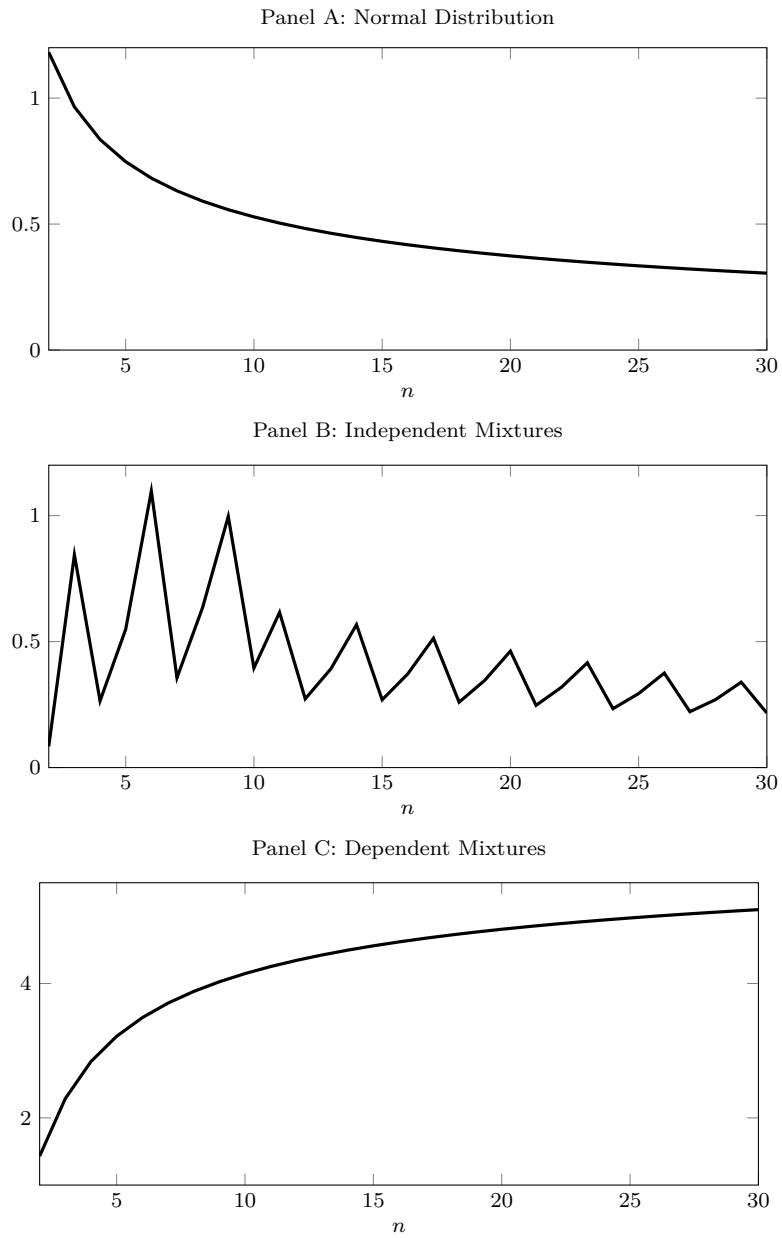
3.4 Policyholders as Owners

We eventually consider the case that the policyholders participating in a pool of size n also provide the risk capital c_n. More specifically, we assume that the policyholders and the equity holders are

[40] Note that this assumption is similar to the distributional requirements in Theorem 5 of Dhaene et al. (2008).

[41] Note that our asymptotic analysis does only cover the case of independent risks. In particular, its implications do thus not apply to the effects illustrated in part iii) of Example 3.

Figure 3: VaR-Based Capital Requirements – Excess Tail Risk



Note: This figure presents the excess tail risk $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ as a function of the size of the risk pool n under VaR-based capital standards for $\alpha = 5\%$ and the distributional assumptions presented in Example 3. Panel A shows results for independent and identically distributed losses with a normal distribution. Panel B is based on independent risks with a two-state normal mixture. Panel C depicts results for dependent risks with a a two-state mixture distribution. We refer to Figure 2 for additional details on the applied distributional assumptions for these examples.

identical and that each of the policyholders provides the same amount of equity capital.

The total payoff that the owners obtain for providing the initial equity capital corresponds to

$$P_n = \max(c_n + \pi\, n - S_n,\, 0). \tag{32}$$

If this payoff is distributed equally, each of the owners receives

$$\bar{P}_n = \frac{1}{n} P_n = \max(\bar{c}_n + \pi - \bar{S}_n, 0) \tag{33}$$

at time $t = 1$ after paying $\bar{c}_n$ at time $t = 0$. The wealth resulting from the combined policyholder and owner position is thus given by

$$W^c_{i,n} = w_{0,i} - \pi - \bar{L}_n - \bar{c}_n + \bar{P}_n. \tag{34}$$

With $\max(a, 0) - \max(-a, 0) = a$, equation (34) simplifies to

$$W^c_{i,n} = w_{0,i} - \bar{S}_n. \tag{35}$$

This exactly corresponds to the position of the policyholders in the case of a mutual insurance company that is in detail analyzed by Gatzert and Schmeiser (2012) and Albrecht and Huggenberger (2017). Therefore, the benefits of larger risk pools documented for this case also apply to the given situation. In particular, we obtain from Theorem 4.1 in Albrecht and Huggenberger (2017) that larger pools are always weakly preferred to smaller pools, i.e., $W^c_{i,n+1} \succeq_{ssd} W^c_{i,n}$, if the policyholders also own the equity stake – irrespective of the premium and the available amount of equity capital. Accordingly, the potential disadvantages of risk pooling for policyholders under a VaR-based regulation are not relevant for mutual insurance companies or, more generally, if the policyholders also own an equity stake.

4 Extensions

We next analyze selected modifications of our baseline assumptions. In Section 4.1, we investigate the potential impact of expenses on the benefits of risk pooling in the presence of default risk.

Section 4.2 introduces investment risk as an additional source of uncertainty that affects default losses. In Section 4.3, we finally analyze the benefits of risk pooling for more general types of coverage and more general rules for sharing the total default loss.

4.1 Expenses

We denote the expenses of an insurer with a risk pool of size n by e_n. Furthermore, $\bar{e}_n := e_n/n$ is used for the average expenses per policyholder. The expenses e_n include operating expenses and capital costs. We assume that these expenses do not grow faster than the size of the risk pool, which captures the decrease of fixed costs per policyholder and the potential impact of diversification benefits on the capital costs. Accordingly, the average expenses per policyholder are weakly decreasing, which we formalize with the following assumption:

Assumption E1 $\bar{e}_n \geq \bar{e}_{n+1}$ *for all* $n \in \mathbb{N}$.

To cover the expenses, policyholders are charged the gross premium

$$\bar{\pi}_n^e = \pi + \bar{e}_n. \tag{36}$$

We thus assume that the expense loading exactly covers the expenses per policyholder. This implies that the expenses do not affect the magnitude of the default loss denoted by $\bar{L}_n^e$. Under our baseline assumptions A2 and A3, it holds that

$$\bar{L}_n^e = \max(\bar{S}_n + \bar{e}_n - \bar{c}_n - \bar{\pi}_n^e, 0) = \max(\bar{S}_n - \bar{c}_n - \pi, 0) = \bar{L}_n. \tag{37}$$

However, under Assumption E1, the gross premium is allowed to vary with the size of the risk pool. This introduces a second channel through which the size of the risk pool affects the utility of policyholders. In particular, the wealth from buying the vulnerable insurance policy accounting for the expense loading is given by

$$W_{i,n}^e = w_{0,i} - \bar{\pi}_n^e - \bar{L}_n^e = W_{i,n} - \bar{e}_n. \tag{38}$$

We can extend our results on the benefits of risk pooling to this wealth position as follows:

Corollary 2 *Suppose that the Assumptions A2, A3 and E1 hold.*

i) If the equity capital is given by $c_n = c\,n$ and Assumption A1 is satisfied, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W^e_{i,n+1} \succeq_{ssd} W^e_{i,n}$ for all $n \geq 1$.

ii) If the equity capital is given by $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$ and Assumption A4 holds, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W^e_{i,n+1} \succeq_{ssd} W^e_{i,n}$ for all $n \geq 1$.

Proof: See the Appendix.

Under volume-based solvency standards, we thus obtain utility gains from risk pooling without additional assumptions. The cost benefits reinforce the preference for larger risk pools that comes from diversification benefits. For the case of a VaR-based capital rule, we again need the excess wealth order condition on the average claim per policyholder. In contrast to the analysis without expenses, we do not obtain an equivalence between this condition and a preference for larger risk pools. This difference reflects that the cost channel can imply a preference for larger pools even if the excess tail risk is not decreasing in the pool size.

Overall, our results confirm the intuition that cost benefits of larger risk pools can be an additional channel that generates utility gains for policyholders.

4.2 Investment Risk

We next investigate the impact of investment risk on our results. We assume that premiums are invested at a random return R and that the profits or losses from this investment reduce or increase the default loss at the end of the period. In contrast, the equity capital (minimum risk capital) may only be invested at the risk-free rate, which is still assumed to be zero. The total default loss for a pool of size n with investment risk is then given by

$$L_n^{inv} = \max(S_n - n\,\pi\,(1+R) - c_n,\, 0) \tag{39}$$

and the average default loss corresponds to

$$\bar{L}_n^{inv} = \max(\bar{S}_n - \pi(1+R) - \bar{c}_n,\, 0). \tag{40}$$

In line with Assumption A3, we consider an equal sharing of the modified default loss among policyholders.

Assumption I1 *With investment risk, the default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = \bar{L}_n^{inv}$ for $i = 1, \ldots, n$.*

Accordingly, the default loss depends on the joint distribution of the average claim per policyholder $\bar{S}_n$ and the investment return R. We introduce the following additional assumptions on the joint distribution of R and the losses X_i, $i = 1, \ldots, n$:

Assumption I2 *The investment return R is independent of the losses $X_1, \ldots, X_n$.*

Assumption I3 *The distribution of the investment return R has a log-concave density.*

Important examples for distributions with log-concave densities include normal distributions and gamma distributions with a shape parameter $\alpha \geq 1$.[42]

For $W_{i,n}^{inv} = w_{0,i} - \pi - \bar{L}_n^{inv}$, we are able to extend the results of our baseline analysis as follows:

Proposition 3 *Suppose that the Assumptions A2, I1 and I2 hold.*

i) If the equity capital is given by $c_n = n\,c$ and Assumption A1 is satisfied, then the utility benefits of risk pooling under default risk are increasing in the pool size, $W_{i,n+1}^{inv} \succeq_{ssd} W_{i,n}^{inv}$ for all $n \geq 1$.

ii) If the equity capital is given by $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi(1+R)]$ and the additional Assumptions A4 and I3 hold, then the utility benefits of risk pooling under default risk are increasing in the pool size, $W_{i,n+1}^{inv} \succeq_{ssd} W_{i,n}^{inv}$ for all $n \geq 1$.

Proof: See the Appendix.

According to part i) of this proposition, independent investment risk does not affect our result that policyholders prefer larger risk pools if the equity capital is proportional to the premiums. If

[42] Cf. Bagnoli and Bergstrom (2005) for a comprehensive discussion of log-concave probability distributions in general and see Table 1 in Bagnoli and Bergstrom (2005) for a list of distributions with log-concave densities.

the equity capital is calculated based on the VaR taking into account the additional investment risk, we do not only need the excess wealth order condition on $\bar{S}_n$ but also an additional restriction on the distribution of the investment return R to establish that larger risk pools are beneficial for policyholders. Furthermore, the conditions stated in ii) of Proposition 3 are only sufficient but not necessary in contrast to the case without investment risk considered in Proposition 2.

4.3 General Coverage and Loss Sharing Rules

We finally show that our main results on the benefits of larger risk pools can also be extended to more general contract types and alternative specifications of the individual loss from default if we replace the exchangeability assumption on the losses with the common assumption of independent and identically distributed losses. We therefore introduce:

Assumption G1 *The losses* $(X_1, \ldots, X_n)$ *are independent and identically distributed with* $\mathbb{E}[|X_i|] < \infty$ *for all* $i = 1, \ldots, n$.

In contrast to Assumption A1, G1 rules out dependent losses, which allows us to separate the impact of the own loss X_i and of the other policyholders' losses X_j, $j \neq i$, on the utility of the policyholder i.

Furthermore, the full-coverage assumption can be relaxed as follows:

Assumption G2 *Agent* i *can buy an insurance contract that pays the indemnity* $f(X_i)$ *for a risk premium* $\pi > 0$, $i = 1, \ldots, n$. *The indemnity function* f *is non-decreasing and convex with* $\mathbb{E}[|f(X_1)|] < \infty$.

In addition to full coverage, this assumption allows for well-studied contracts such as linear coverage (Mossin, 1968) or contracts with deductibles (Arrow, 1974). Assumption G2 implies that the type of coverage, i.e. the indemnity function, is identical for all policies. In combination with Assumption G1, this ensures that we are again analyzing a homogeneous pool.

To generalize Assumption A3, we assume that the individual default loss $D_{i,n}$ depends on the own claim $f(X_i)$ and the average claim of the other policyholders

$$\bar{S}^f_{i,n} = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f(X_j). \tag{41}$$

Specifically, we introduce

$$\bar{L}^f_{i,n} = \max(\bar{S}^f_{i,n} - \pi - \bar{c}_{i,n},\, 0) \tag{42}$$

as the average default loss of the risk pool without policyholder i, where $\bar{c}_{i,n}$ is the equity capital for the pool without policy i. This definition implicitly assumes that the pool is large and granular enough so that the claim of policyholder i is not relevant for the default of the portfolio. In other words, adding policy i to the pool does not change the occurrence of the default event. Nevertheless, the policyholder's claim can be relevant for her loss in case of a default as described by the following general default loss specification.

Assumption G3 *The default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = g(\bar{L}^f_{i,n}, f(X_i))$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$, with a measurable function $g : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $g(0, y) = 0$ for $y \in \mathbb{R}$. g is (weakly) increasing and convex in both arguments.*

A central feature of this definition is that the individual default loss is increasing in both, the own claim and the average total default loss of the other policyholders. Furthermore, the requirement $g(0, y) = 0$ for all $y \in \mathbb{R}$ ensures that the individual default loss is zero if there is no excess loss for the other policyholders, i.e., $\bar{L}^f_{i,n} = 0$. According to the convexity of g, the increase in $D_{i,n}$ is higher for high levels of $\bar{L}^f_{i,n}$ or $f(X_i)$. This allows for lower levels of excess losses to be partially covered by additional reserves or a guaranty fund.[43]

We briefly discuss examples for specific default functions g that satisfy Assumption G3:

Example 5 *Similar to the sharing rule used in our baseline analysis, an equal distribution of default losses corresponds to the function $g(l, y) := \beta\, l$ for $\beta \geq 0$. With this choice, we obtain*

$$D_{i,n} = \beta\, \bar{L}^f_{i,n} \tag{43}$$

for the default loss of policyholder i. A simple extension of this rule that also considers the magnitude of the own claim amount is $g(l, y) = \beta\, l \max(y, 0)$, where again $\beta \geq 0$.[44] *The individual*

[43] The idea of introducing general "sharing rules" for the default loss is already implicit in Condition 1 of Ibragimov et al. (2010). Compared to their conditions, we do not impose an upper bound on the individual share of the default loss and we do not explicitly require that the individual default losses add up to the total excess loss. Note that the latter requirement might be less appropriate in our setting as it rules out a partial compensation of the policyholders by a guaranty fund.

[44] The max-function is not required if the loss distribution and the indemnity function imply that $f(X_i)$ is nonnegative, which is typically the case.

default loss is then given by

$$D_{i,n} = \beta\, \bar{L}^f_{i,n} \max(f(X_i),\, 0). \tag{44}$$

Accordingly, policyholders with a high claim are assigned a larger fraction of the total default loss. Note that similar rules are commonly used in insurance bankruptcy laws.[45]

Under the Assumptions G2 and G3, the wealth from buying a vulnerable insurance contract corresponds to

$$W^g_{i,n} = w_{0,i} - \pi - X_i + f(X_i) - g(\bar{L}^f_{i,n}, f(X_i)). \tag{45}$$

For this wealth position, we can establish the following results:

Proposition 4 *Suppose that the Assumptions G1, G2 and G3 hold.*

i) If the average equity capital is constant, i.e. $\bar{c}_{i,n} = c$*, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W^g_{i,n+1} \succeq_{ssd} W^g_{i,n}$ *for all* $n \geq 1$*.*

ii) If the average equity capital is given by $\bar{c}_{i,n} = \mathrm{VaR}_\alpha\left[\bar{S}^f_{i,n} - \pi\right]$ *and* $\bar{S}^f_{i,n+1} \preceq_{ew} \bar{S}^f_{i,n}$ *holds for all* $n \geq 1$*, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W^g_{i,n+1} \succeq_{ssd} W^g_{i,n}$*.*

Proof: See the Appendix.

According to Proposition 4, the results of our baseline analysis can be extended to more general contract types and to alternative sharing rules for the total default loss. Under a capital regulation with a proportional growth of the reserves, Assumption G1 is sufficient for obtaining utility gains from insurance in larger risk pools. To obtain the same results under a VaR-based capital regulation, we again need an excess-wealth-order requirement that is analogous to Assumption A4.

5 Concluding Remarks

We investigate the consequences of risk pooling from the policyholders' perspective under different solvency rules. We focus on the case of stock insurance companies taking into account the limited

[45] See, e.g., the U.S. Insurer Receivership Model Act, which states that "all claims allowed within a priority class shall be paid at substantially the same percentage" (National Association of Insurance Commissioners, 2007, Section 802).

liability of equity holders. For our analysis, we assume exchangeable risks and apply an endogenous default definition. In addition, we use an SSD criterion for utility comparisons and assume that the equity capital is exogenously determined with volume-based or VaR-based capital requirements.

Under these rather general assumptions, risk pooling within a stock insurance company can affect the policyholder's position through a reduction (or an increase) of utility losses from default risk. If the equity capital grows proportionally with the pool size, all risk-averse policyholders benefit from risk pooling without additional assumptions. In contrast, we demonstrate that pooling a larger number of identical policies can adversely affect the policyholders' utility from insurance under a VaR-based regulation. For this case, we show that policyholders attain utility gains from larger risk pools if and only if the average claim's excess tail risk, as measured by the difference between AVaR and VaR, does not increase with the pool size.

Our analysis clarifies the effects of risk pooling on the policyholders' utility beyond the mutual insurance case analyzed by Albrecht and Huggenberger (2017). Furthermore, our findings indicate that it can be important to complement well-known asymptotic techniques with results for finite n to understand the economic implications of risk pooling and diversification. In particular, our non-asymptotic results reveal that diversification benefits on the portfolio level can have unexpected adverse consequences for policyholders under a VaR-based solvency framework.

An interesting direction for future research can be the analysis of risk pooling from the policyholders' perspective under alternative regulatory frameworks or different assumptions that determine the available amount of equity for a given pool size.

Appendix

We first summarize a selection of results on SSD, increasing convex order and excess wealth order, which we will use in the following proofs.

Lemma 2

i) Let X and Y be two random variables. Then

$$X \preceq_{icx} Y \qquad \Leftrightarrow \qquad -X \succeq_{ssd} -Y. \tag{46}$$

ii) Let f (g) be an increasing convex (concave) function. Then, it holds for the random variables X and Y that

$$X \preceq_{icx} Y \qquad \Rightarrow \qquad f(X) \preceq_{icx} f(Y), \tag{47}$$

$$X \preceq_{ssd} Y \qquad \Rightarrow \qquad g(X) \preceq_{ssd} g(Y). \tag{48}$$

iii) Let (X_1, X_2) and (Y_1, Y_2) be pairs of independent random variables. If $X_i \preceq_{ssd} Y_i$, $i = 1, 2$, and $w : \mathbb{R}^2 \to \mathbb{R}$ is component-wise increasing and concave, then

$$w(X_1, X_2) \preceq_{ssd} w(Y_1, Y_2). \tag{49}$$

If $X_i \preceq_{icx} Y_i$, $i = 1, 2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is component-wise increasing and convex, then

$$g(X_1, X_2) \preceq_{icx} g(Y_1, Y_2). \tag{50}$$

iv) Let X and Y be two random variables. Then

$$X \preceq_{ew} Y \quad \Leftrightarrow \quad \max(X - Q_\alpha[X], 0) \preceq_{icx} \max(Y - Q_\alpha[Y], 0) \quad \textit{for all } \alpha \in (0, 1). \tag{51}$$

v) Let X and Y be random variables with $X \preceq_{ew} Y$ and let Z be a random variable that is

independent of X and Y. If Z has a log-concave density, then

$$X + Z \preceq_{ew} Y + Z. \tag{52}$$

See Shaked and Shanthikumar (2007, Theorem 4.A.1) for the result in i). Part ii) and the icx-result in part iii) can be found in the Theorems 4.A.8 and 4.A.15 in Shaked and Shanthikumar (2007). The equivalence stated in equation (51) is shown by Sordo (2008, Theorem 6ii).[46] v) follows from Theorem 3.1 in Hu et al. (2006) and the relationship between excess wealth order and location independent risk order.

Proof of Proposition 1: Given Assumption A1, Lemma 1 and Lemma 2 i) imply for the average claim amount per policyholder that

$$\bar{S}_{n+1} \preceq_{icx} \bar{S}_n. \tag{53}$$

Since the function $\psi(x) = \max(x - c - \pi, 0)$ is increasing and convex, it follows from Lemma 2 ii) that

$$\bar{L}^c_{n+1} = \psi(\bar{S}_{n+1}) \preceq_{icx} \psi(\bar{S}_n) = \bar{L}^c_n. \tag{54}$$

Using equation (46), we conclude

$$-\bar{L}^c_{n+1} \succeq_{ssd} -\bar{L}^c_n. \tag{55}$$

Note that the function ψ defined above is not strictly increasing. Therefore, we cannot establish results based on a strict version of second order stochastic dominance with this reasoning.

For $\eta(x) := w_{0,i} - \pi + x$, it holds that $W_{i,n} = \eta(\bar{L}^c_n)$ and $W_{i,n+1} = \eta(\bar{L}^c_{n+1})$. Since η is an increasing linear transformation, $W_{i,n+1} \succeq_{ssd} W_{i,n}$ follows from equation (48) in Lemma 2. ■

Proof of Proposition 2: Due to the representation in equation (24), Lemma 2 iv) implies that

$$\bar{L}^v_{n+1} \preceq_{icx} \bar{L}^v_n \qquad \Leftrightarrow \qquad \bar{S}_{n+1} \preceq_{ew} \bar{S}_n. \tag{56}$$

[46] A version of this result for continuous distributions can also be found in Shaked and Shanthikumar (2007, Theorem 4.A.43).

Using Lemma 2 i), we conclude

$$-\bar{L}^v_{n+1} \succeq_{ssd} -\bar{L}^v_n \qquad \Leftrightarrow \qquad \bar{S}_{n+1} \preceq_{ew} \bar{S}_n. \tag{57}$$

Furthermore, note that $W_{i,n} = \eta(-\bar{L}_n)$ and $-\bar{L}_n = \eta^{-1}(W_{i,n})$ with $\eta(x) = w_{0,i} - \pi + x$ and $\eta^{-1}(y) = y - w_{0,i} + \pi$. Since both functions are increasing and linear (and thus concave), it follows from equations (48) and (57) that

$$W_{i,n+1} \succeq_{ssd} W_{i,n} \qquad \Leftrightarrow \qquad \bar{S}_{n+1} \preceq_{ew} \bar{S}_n. \tag{58}$$

■

Proof of Corollary 1: We have to show that the Assumptions A1 and A5 imply Assumption A4, then the result follows from Proposition 2. Given that $\mathbb{E}\big[Z^2\big] < \infty$, we can define $Z^* := \frac{Z - \mathbb{E}[Z]}{\sigma[Z]}$. Then, it holds that $S_n \overset{\mathrm{d}}{=} \mathbb{E}[S_n] + \sigma[S_n] \cdot Z^*$ for all $n \in \mathbb{N}$. Due to the translation invariance and the positive homogeneity of VaR and AVaR, this implies

$$\mathrm{VaR}_\alpha\Big[\bar{S}_n\Big] = \mathbb{E}\Big[\bar{S}_n\Big] + \sigma[\bar{S}_n] \cdot \mathrm{VaR}_\alpha[Z^*]\,, \tag{59}$$

$$\mathrm{AVaR}_\alpha\Big[\bar{S}_n\Big] = \mathbb{E}\Big[\bar{S}_n\Big] + \sigma[\bar{S}_n] \cdot \mathrm{AVaR}_\alpha[Z^*]\,. \tag{60}$$

We obtain

$$\mathrm{AVaR}_\alpha\Big[\bar{S}_n\Big] - \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big] = \sigma[\bar{S}_n]\,\left(\mathrm{AVaR}_\alpha[Z^*] - \mathrm{VaR}_\alpha[Z^*]\right). \tag{61}$$

Therefore, Assumption A4 is satisfied due to equation (30), $\mathrm{AVaR}_\alpha[Z^*] \geq \mathrm{VaR}_\alpha[Z^*]$ and $\sigma[\bar{S}_{n+1}] \leq \sigma[\bar{S}_n]$. To see that the latter inequality holds, we note that Assumption A1 implies $\sigma[X_i] = \sigma$ for all $i = 1, \ldots, n$ and $\rho[X_i, X_j] = \rho$ for $i \neq j$. We thus have

$$\mathrm{var}\Big[\bar{S}_n\Big] = \mathrm{var}\Big[\frac{1}{n} S_n\Big] = \frac{1 + (n-1)\rho}{n}\sigma^2 \tag{62}$$

and therefore $\mathrm{var}\Big[\bar{S}_{n+1}\Big] \leq \mathrm{var}\Big[\bar{S}_n\Big]$ due to $\rho \leq 1$. ■

Proof of Corollary 2: Under Assumption E1, it holds that $-\bar{e}_{n+1} \succeq_{ssd} -\bar{e}_n$. The remaining assumptions in Corollary 2 imply $W_{i,n+1} \succeq_{ssd} W_{i,n}$ according to Proposition 1 for i) and according

to Proposition 2 for ii). We can thus use equation (50) from Lemma 2 with $w(x_1, x_2) = x_1 + x_2$ to conclude

$$W^e_{i,n+1} = W_{i,n+1} - \bar{e}_{n+1} \succeq_{ssd} W_{i,n} - \bar{e}_n = W^e_{i,n}. \tag{63}$$

■

Proof of Proposition 3: We introduce $\bar{S}^{inv}_n := \bar{S}_n - \pi R$ for all n. Then, the average default loss with investment proceeds can be written as

$$\bar{L}^{inv}_n = \max(\bar{S}^{inv}_n - \pi - \bar{c}_n, 0). \tag{64}$$

For the case in i), we exploit that $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ implies

$$\bar{S}^{inv}_{n+1} \preceq_{icx} \bar{S}^{inv}_n. \tag{65}$$

This follows from the independence of $\bar{S}_n$ and $-\pi R$ and the closure property stated in equation (50) of Lemma 2 with $g(x_1, x_2) = x_1 + x_2$. Given equations (64) and (65), the preference for larger risk pools follows using the same arguments as in the proof of Proposition 1.

For the case in ii), we first note that log-concavity of the density of R implies that the density of $-\pi R$ is also log-concave (Bagnoli and Bergstrom, 2005, Corollary 5 and Theorem 8). With the Assumptions I2 and I3, we obtain

$$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n \qquad \Rightarrow \qquad \bar{S}^{inv}_{n+1} \preceq_{ew} \bar{S}^{inv}_n \tag{66}$$

from the definition of $\bar{S}^{inv}_n$ and the stability result for excess wealth order under convolutions as stated in Lemma 2 v). Moreover, it holds that

$$\bar{L}^{inv}_n = \max(\bar{S}_n - \pi R - Q_{1-\alpha}[\bar{S}_n - \pi R], 0) = \max(\bar{S}^{inv}_n - Q_{1-\alpha}[\bar{S}^{inv}_n], 0). \tag{67}$$

Using $\bar{S}^{inv}_{n+1} \preceq_{ew} \bar{S}^{inv}_n$, the SSD-ordering of the corresponding wealth positions can be derived as in the proof of Proposition 2. ■

Proof of Proposition 4: Under the Assumptions G1 and G2, $(f(X_1), \ldots, f(X_n))$ is a collection of independent and identically distributed random variables with $\mathbb{E}[|f(X_i)|] < \infty$ for all

$i = 1, \ldots, n$. Therefore, we obtain from Lemma 1 that $\bar{S}^f_{i,n+1} \preceq_{icx} \bar{S}^f_{i,n}$.

We first consider the case in i), i.e. $\bar{c}_{i,n} = c$: Then, $\psi(x) = \max(x - \pi - c,\, 0)$ is non-decreasing and convex. Therefore, $\bar{L}^f_{i,n}$ defined in equation (42) satisfies $\bar{L}^f_{i,n+1} = \psi(\bar{S}^f_{i,n+1}) \preceq_{icx} \psi(\bar{S}^f_{i,n}) = \bar{L}^f_{i,n}$. With $\bar{L}^*_{i,n} = -\bar{L}^f_{i,n}$, Lemma 2 i) implies

$$\bar{L}^*_{i,n+1} = -\bar{L}^f_{i,n+1} \succeq_{ssd} -\bar{L}^f_{i,n} = \bar{L}^*_{i,n}. \tag{68}$$

Next, we introduce the function $h_i : \mathbb{R}^2 \to \mathbb{R}$ with

$$h_i(l^*, x^*) = w_{0,i} + x^* - f(-x^*) - \pi - g(-l^*, f(-x^*)). \tag{69}$$

From equation (45) and this definition, it follows that $W^g_{i,n} = h_i(\bar{L}^*_{i,n}, X^*_i)$ with $X^*_i = -X_i$. Due to Assumption G3, h_i is increasing and concave in l^*. From the Assumptions G2 and G3, it follows that h_i is also increasing and concave in x^*.

By construction, X^*_i and $\bar{L}^*_{i,n}$ are independent. Using $X^*_i \succeq_{ssd} X^*_i$ and equation (68), we conclude from Lemma 2 iii) that

$$W^g_{i,n+1} = h_i(\bar{L}^*_{i,n+1}, X^*_i) \succeq_{ssd} h_i(\bar{L}^*_{i,n}, X^*_i) = W^g_{i,n}. \tag{70}$$

Next, we turn to the case ii), i.e., $\bar{c}_{i,n} = \mathrm{VaR}_\alpha\left[\bar{S}^f_{i,n} - \pi\right]$: In this case, it holds that

$$\bar{L}^f_{i,n} = \max(\bar{S}^f_{i,n} - \pi - \mathrm{VaR}_\alpha\left[\bar{S}^f_{i,n} - \pi\right],\, 0) = \max(\bar{S}^f_{i,n} - Q_{1-\alpha}[\bar{S}^f_{i,n}],\, 0). \tag{71}$$

From Lemma 2 iv) and equation (71), we conclude

$$\bar{S}^f_{i,n+1} \preceq_{ew} \bar{S}^f_{i,n} \qquad \Leftrightarrow \qquad \bar{L}^f_{i,n+1} \preceq_{icx} \bar{L}^f_{i,n}. \tag{72}$$

To prove the preference for larger risk pools, we can proceed as in part i). ■

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Online Appendix

Risk Pooling and Solvency Regulation: A Policyholder's Perspective

Abstract: This Online Appendix consists of four sections. In Section I, we provide an asymptotic analysis on the benefits of risk pooling for the policyholders of stock insurance companies. Section II presents required results on (partial) moment generating functions. In Section III, we provide additional details on the examples included in our main analysis. Section IV contains an additional example with a simple discrete loss distribution.

I Asymptotic Results

We investigate the asymptotic impact of default risk on the policyholders' utility in a setting that is similar to our baseline analysis in Section 3.

For the specification of the contract type and the individual default loss, we again rely on the Assumptions A2 and A3. However, we focus on the standard expected utility model ($g_i(p) = p$) and we replace Assumption A1 with the following conditions on the loss distribution and the utility function.

Assumption L1 *The losses $(X_i)_{i\in\mathbb{N}}$ are identically distributed and independent with $0 \leq X_i \leq M < \infty$, $0 \leq \pi \leq M$ and $w_{0,i} \geq 0$ for all $i = 1, \ldots, n$. u_i is continuous on $[-2\,M, w_{0,i}]$ for all $i = 1, \ldots, n$.*

Accordingly, we consider IID instead of exchangeable risks and we additionally assume that these risks are bounded, which is a common restriction in the economic risk analysis that avoids convergence problems of limited economic interest (Rothschild and Stiglitz, 1970, p. 227).[1] In the "theory of insurance demand", the maximum loss M is often set to the initial wealth of a representative agent to exclude bankruptcy issues on the level of the individual policyholders (Schlesinger, 2013, p. 168). The boundedness of the premium can be seen as a direct consequence of $X_i \leq M$. The last part of the assumption ensures that the utility function only takes finite values.

For our asymptotic analysis, we do not need to specify the exact dependence of the equity capital on n. Instead we only assume that the average capital per policyholder converges to a limit with $\lim_{n\to\infty} \bar{c}_n = c_a < \infty$.

Building on these assumptions, we can establish the following asymptotic result:

Proposition I.1 *Suppose that the Assumptions L1, A2 and A3 hold and that $c_a \geq \mathbb{E}[f(X_i)] - \pi$. Then, the expected utility of the vulnerable contract converges to the utility of the safe insurance contract, i.e.,*

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = u(W_i^s). \tag{I.1}$$

[1] In particular, the boundedness of the losses allows us to invoke Lebesgue's dominated convergence theorem for interchanging integration and almost sure convergence, which requires the existence of an integrable dominating function.

Proof: We first note that a strong law of large numbers applies to $(X_i)_{i\in\mathbb{N}}$ under Assumption L1 (see, e.g., Theorem 5.17 in Klenke, 2014). Therefore, the average total claim amount converges to the expected claim amount, i.e.,

$$\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mathbb{E}[X_i]\,. \tag{I.2}$$

Second, the continuity of $x \mapsto \max(x - \pi - \bar{c}_n,\, 0)$ implies

$$\bar{L}_n = \max(\bar{S}_n - \pi - \bar{c}_n,\, 0) \xrightarrow{\text{a.s.}} \max(\mathbb{E}[X_i] - \pi - c_a,\, 0) = 0. \tag{I.3}$$

For the last equality, we use that $\lim_{n\to\infty} \bar{c}_n = c_a \geq \mathbb{E}[X_i] - \pi$. From equation (I.3), it follows that

$$W_{i,n} = w_{0,i} - \pi - \bar{L}_n \xrightarrow{\text{a.s.}} w_{0,i} - \pi = W_i^s. \tag{I.4}$$

Since u_i is continuous, the utility of the risky contract satisfies $u_i(W_{i,n}) \xrightarrow{\text{a.s.}} u_i(W_i^s)$.

Third, we can establish integrable bounds for $|u_i(W_{i,n})|$. Assumption L1 implies $0 \leq \bar{L}_n \leq \bar{S}_n \leq M$ and $-M \leq w_{0,i} - \pi \leq w_{0,i}$. From combining these inequalities, it follows that

$$-2M \leq W_{i,n} \leq w_{0,i}. \tag{I.5}$$

Since u_i is assumed to be continuous on this interval according to Assumption L1, the extreme value theorem implies that u_i is bounded on $[-2M, w_{0,i}]$. Therefore, we can invoke Lebesgue's convergence Theorem (Klenke, 2014, Corollary 6.26) to conclude

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = \mathbb{E}\Big[\lim_{n\to\infty} u_i(W_{i,n})\Big] = u_i(W_i^s), \tag{I.6}$$

which corresponds to (I.1). ■

Accordingly, the disutility from the default loss asymptotically goes to zero and the expected utility of buying the vulnerable contract reaches the level of the safe insurance contract for large pools.

If the equity capital increases proportionally with the pool size, i.e., if $\bar{c}_n = c$ for all $n \in \mathbb{N}$,

then $c_a \geq \mathbb{E}[X_i] - \pi$ is obviously equivalent to $c \geq \mathbb{E}[X_i] - \pi$. This result can be illustrated by reconsidering Example 2.

Example I.1 *Under the assumptions made in Example 2, the expected utility of buying a vulnerable insurance contract is given by equations* (22) *and* (23)*. Furthermore,* $c_a \geq \mathbb{E}[f(X_i)] - \pi$ *from Proposition I.1 implies* $c \geq \mu - \pi$ *and thus* $l \geq 0$*. Using these results, it is not difficult to show that* $\lim_{n\to\infty} \mathbb{M}_{\bar{L}_n^c}(\gamma) = 1$ *and thus*

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = 1 - \exp(-\gamma\,(w_{0,i} - \pi)) = \mathbb{E}[u_i(W_i^s)]\,. \tag{I.7}$$

This convergence was illustrated in Panel B of Figure 1 in the main text.

Since the normal distribution does not satisfy the boundedness restriction from Assumption L1, this example shows that the boundedness of X_i is only sufficient but not necessary for the applicability of limit arguments similar to Proposition I.1.

Proposition I.1 also applies to VaR-based solvency standards. In contrast to our finite sample results, the additional excess wealth order condition is not required. To illustrate the convergence for the VaR-based case, we again consider independent normally distributed losses and an exponential utility function.

Example I.2 *Under the assumptions presented in part i) of Example 3, we can explicitly compute the limit of the expected utility from the results in equations* (25) *and* (26)*. We obtain*

$$\lim_{n\to\infty} \mathbb{M}_{\bar{L}_n^v}(\gamma) = \exp(0)\,\left[1 - \Phi\left(\Phi^{-1}(1-\alpha)\right)\right] + (1-\alpha) = 1 \tag{I.8}$$

and therefore again $\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = \mathbb{E}[u_i(W_i^s)]$*. This convergence was illustrated in Panel A of Figure 2 in the main text.*

In this context, it is important to note that our asymptotic results only apply to the case of independent risks. They are thus not applicable to the setting considered in part iii) of Example 3, where we documented a monotonic decrease of the policyholders' utility under a VaR-based regulation.

Finally, note that Proposition I.1 can be extended to more general contract types and specifications of the individual default loss. In particular, a generalized version of the convergence result can be obtained for the setting studied in Section 4.3 under suitable continuity and boundedness restrictions on the functions f and g from the Assumptions G2 and G3.[2]

II Selected Results on Moment Generating Functions

The moment generating function of a random variable X is defined as $\mathbb{M}_X(\gamma) = \mathbb{E}[\exp(\gamma\, X)]$. If X is normally distributed with expected value μ and variance σ^2, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$, it is well known that

$$\mathbb{M}_X(\gamma) = \exp\left(\gamma\,\mu + \frac{1}{2}\,\gamma^2\,\sigma^2\right). \tag{II.1}$$

If X follows a mixture of normals with K components, we can introduce a state indicator Y with $X|Y = k \sim \mathcal{N}(\mu_k, \sigma_k^2)$ and $\mathbb{P}[Y = k] = p_k$ for $k = 1, \ldots, K$. Then, the moment generating function of X can be written as

$$\mathbb{M}_X(\gamma) = \sum_{k=1}^{K} \mathbb{E}[\exp(\gamma\, X) \mid Y = k]\ \mathbb{P}[Y = k] \tag{II.2}$$

$$= \sum_{k=1}^{K} p_k\ \exp\left(\mu_k\,\gamma + \frac{1}{2}\gamma^2\sigma_k^2\right). \tag{II.3}$$

We define the (upper) partial moment generating function of X as

$$\mathbb{M}_{X,t}(\gamma) = \mathbb{E}[\exp(\gamma\, X)\mathbb{1}(X \geq t)]\,. \tag{II.4}$$

If again $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\mathbb{M}_{X,t}(\gamma) = \int_t^\infty \exp(\gamma\, x)\frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right) dx \tag{II.5}$$

$$= \int_{\tilde{t}}^\infty \exp(\gamma\,\mu)\ \exp(\gamma\,\sigma\, z)\,\varphi(z)dz \tag{II.6}$$

$$= \int_{\tilde{t}}^\infty \exp\left(\gamma\,\mu + \frac{1}{2}\sigma^2\gamma^2\right)\ \varphi(z - \gamma\,\sigma)dz \tag{II.7}$$

$$= \mathbb{M}_X(\gamma)\left[1 - \Phi\left(\frac{t-\mu}{\sigma} - \gamma\cdot\sigma\right)\right]. \tag{II.8}$$

[2]Results are available from the authors on request.

For the case of a normal mixture with K components, this implies

$$\mathbb{M}_{X,t}(\gamma) = \sum_{k=1}^{K} \mathbb{E}[\exp(\gamma\, X)\, \mathbb{1}(X \geq t) \mid Y = k]\; \mathbb{P}[Y = k] \tag{II.9}$$

$$= \sum_{k=1}^{K} p_k\, \exp\left(\mu_k\, \gamma + \frac{1}{2}\gamma^2 \sigma_k^2\right) \left(1 - \Phi\left(\frac{t - \mu_k}{\sigma_k} - \gamma\, \sigma_k\right)\right). \tag{II.10}$$

III Details Examples

Details Example 2: To derive equation (23), we first note that the moment-generating function of $\bar{L}_n = \max(\bar{S}_n - \pi - \bar{c}_n, 0)$ can be decomposed as follows

$$\mathbb{M}_{\bar{L}_n}(\gamma_i) = \mathbb{E}\Big[\exp\left(\gamma_i\, \bar{L}_n\right) \mathbb{1}\left(\bar{L}_n > 0\right)\Big] + \mathbb{E}\Big[\exp\left(\gamma_i\, \bar{L}_n)\right)\; \mathbb{1}\left(\bar{L}_n \leq 0\right)\Big] \tag{III.1}$$

$$= \exp(-\gamma_i\, (\pi + \bar{c}_n))\, \mathbb{E}\Big[\exp\left(\gamma_i\, \bar{S}_n\right) \mathbb{1}(\bar{S}_n > \pi + \bar{c}_n)\Big] + \mathbb{P}\Big[\bar{S}_n \leq \pi + \bar{c}_n\Big] \tag{III.2}$$

$$= \exp(-\gamma_i\, (\pi + \bar{c}_n))\, \mathbb{M}_{\bar{S}_n, \pi + \bar{c}_n}(\gamma_i) + \mathbb{P}\Big[\bar{S}_n \leq \pi + \bar{c}_n\Big], \tag{III.3}$$

where $\mathbb{M}_{\bar{S}_n, \pi + \bar{c}_n}$ denotes the upper partial moment-generating function of $\bar{S}_n$ for the threshold $t = \pi + \bar{c}_n$. With $\bar{S}_n \sim \mathcal{N}(\mu, \frac{1}{n}\sigma^2)$, it holds that

$$\mathbb{P}\Big[\bar{S}_n \leq \pi + \bar{c}_n\Big] = \mathbb{P}\left[Z \leq \frac{\pi + \bar{c}_n - \mu}{\frac{\sigma}{\sqrt{n}}}\right] = \Phi\left(\frac{\sqrt{n}\, l_n}{\sigma}\right), \tag{III.4}$$

where $l_n = \pi + \bar{c}_n - \mu$. For the computation of $\mathbb{M}_{\bar{S}_n, \pi + \bar{c}_n}(\gamma_i)$, we use equation (II.8) with

$$\frac{\pi + \bar{c}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}\, (\pi + \bar{c}_n - \mu)}{\sigma} = \frac{\sqrt{n}\, l_n}{\sigma} \qquad \text{and} \qquad -\gamma_i\, \sigma[\bar{S}_n] = -\gamma_i\, \frac{\sigma}{\sqrt{n}}. \tag{III.5}$$

This implies

$$\mathbb{M}_{\bar{L}_n}(\gamma_i) = \exp\left(-\gamma_i\, l_n + \frac{1}{2}\, \gamma_i^2\, \frac{\sigma^2}{n}\right) \cdot \left[1 - \Phi\left(\frac{\sqrt{n}\, l_n}{\sigma} - \gamma_i\, \frac{\sigma}{\sqrt{n}}\right)\right] + \Phi\left(\frac{\sqrt{n}\, l_n}{\sigma}\right). \tag{III.6}$$

Using $l = \pi + c - \mu$, we obtain the representation of $\mathbb{M}_{\bar{L}_n^c}(\gamma_i)$ given in equation (23). For the derivative of $\mathbb{E}[u_i(W_{i,n})]$, it then holds that

$$\frac{d\mathbb{E}[u_i(W_{i,n})]}{dn} = -\exp\left(-\gamma_i\, (w_{0,i} - \pi)\right)\, \frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} \tag{III.7}$$

and

$$\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} = \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\,\left\{\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2}\cdot(-1)\cdot\left[1-\Phi\left(\frac{\sqrt{n}\,l}{\sigma}-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]\right.$$
$$\left. + (-1)\cdot\varphi\left(\frac{\sqrt{n}\,l}{\sigma}-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\cdot\frac{1}{2}\,\gamma_i\,\sigma n^{-3/2}\right\}$$
$$+\frac{l}{2\sqrt{n}\,\sigma}\left[\varphi\left(\frac{\sqrt{n}\,l}{\sigma}\right)-\exp\left(-\gamma_i\, l+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\cdot\varphi\left(\frac{\sqrt{n}\,l}{\sigma}-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]. \tag{III.8}$$

The last term vanishes because of

$$\exp\left(-\gamma_i\, l+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\cdot\varphi\left(\frac{\sqrt{n}\,l}{\sigma}-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)=\varphi\left(\frac{\sqrt{n}\,l}{\sigma}\right). \tag{III.9}$$

Therefore, it follows that

$$\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn}\leq 0 \qquad \text{and thus} \qquad \frac{d\mathbb{E}[u_i(W_{i,n})]}{dn}\geq 0. \tag{III.10}$$

Details Example 3 i): The moment-generating function of $\bar{L}_n^v$ given in equation (26) can easily be derived from equation (III.6) with

$$l_n = \mathrm{VaR}_\alpha\left[\bar{S}_n\right]-\mu = \mu+\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)-\mu = \frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha).$$

We therefore obtain

$$\mathbb{M}_{\bar{L}_n^v}(\gamma_i) = \exp\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)$$
$$\cdot\left[1-\Phi\left(\Phi^{-1}(1-\alpha)-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]+(1-\alpha).$$

For the derivative of this function with respect to n, it holds that

$$\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn} = \exp\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \tag{III.11}$$
$$\cdot\left\{\left(\frac{1}{2}\gamma_i\sigma\, n^{-3/2}\,\Phi^{-1}(1-\alpha)-\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2}\right)\cdot\left[1-\Phi\left(\Phi^{-1}(1-\alpha)-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]\right.$$
$$\left.-\,\varphi\left(\Phi^{-1}(1-\alpha)-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\,\frac{1}{2}\,\gamma_i\,\sigma\, n^{-3/2}\right\}.$$

With $\tilde{z} = \Phi^{-1}(1-\alpha) - \gamma_i \frac{\sigma}{\sqrt{n}}$, this can be rewritten as

$$\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn} = \exp\left(-\gamma_i \frac{\sigma}{\sqrt{n}} \Phi^{-1}(1-\alpha) + \frac{1}{2}\gamma_i^2 \frac{\sigma^2}{n}\right) \cdot \frac{1}{2}\gamma_i \sigma n^{-3/2} \left(\tilde{z}\left(1-\Phi(\tilde{z})\right) - \varphi(\tilde{z})\right). \tag{III.12}$$

We next show that

$$\tilde{z}\left(1-\Phi(\tilde{z})\right) < \varphi(\tilde{z}). \tag{III.13}$$

Therefore, we first note that $\varphi'(x) = -x\,\varphi(x) = -x\,\Phi'(x)$ and

$$\varphi(x) = -\int_{-\infty}^{x} s\,\Phi'(s)\,ds = -\left([s\,\Phi(s)]_{-\infty}^{x} - \int_{-\infty}^{x} \Phi(s)\,ds\right). \tag{III.14}$$

With $\lim_{s\to-\infty} s\,\Phi(s) = 0$, this implies

$$\varphi(x) + x\,\Phi(x) = \int_{-\infty}^{x} \Phi(s)\,ds > 0. \tag{III.15}$$

From $\varphi(x) = \varphi(-x)$ and $\Phi(x) = (1-\Phi(-x))$, we obtain

$$\varphi(-x) > -x\,(1-\Phi(-x)). \tag{III.16}$$

Choosing $x = -\tilde{z}$ proves equation (III.13) and thus $\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn} \leq 0$.

Details Example 3 ii): To derive the distribution of S_n under the mixture assumption, we rely on the following characterization of a normal mixture. Let $Y_1, \ldots, Y_n$ denote a series of state variables that are independent and identically distributed with $\mathbb{P}[Y_i = L] = p_L$ and $\mathbb{P}[Y_i = H] = p_H$ for all $i \in \mathbb{N}$, where $p_L + p_H = 1$. Furthermore, the distribution of X_i only depends on Y_i (X_i is independent of Y_j, $j \neq i$) and the conditional distribution of X_i is given by

$$X_i | Y_i = L \sim \mathcal{N}(\mu_L, \sigma_L^2) \qquad \text{and} \qquad X_i | Y_i = H \sim \mathcal{N}(\mu_H, \sigma_H^2). \tag{III.17}$$

X_i is thus normally distributed conditional on Y_i with state-specific mean and variance parameters

(μ_L, σ_L^2) and (μ_H, σ_H^2). Under these assumption the unconditional distribution of X_i satisfies

$$\mathbb{P}[X_i \leq x] = p_L\, \Phi(x, \mu_L, \sigma_L^2) + p_H\, \Phi(x, \mu_H, \sigma_H^2). \tag{III.18}$$

We next introduce an indicator for the occurrence of the high loss state

$$H_i := \mathbb{1}(Y_i = H) = \begin{cases} 0 & \text{if } Y_i = L \\ 1 & \text{if } Y_i = H \end{cases} \tag{III.19}$$

and the counting variable $C_n = \sum_{i=1}^n H_i$. Then, H_i is a Bernoulli random variable with $\mathbb{P}[H_i = 1] = \mathbb{P}[Y_i = H] = p_H$ and C_n follows a Binomial distribution. This implies

$$\mathbb{P}[C_n = k] = \binom{n}{k} p_H^k\, (1 - p_H)^{n-k} = \binom{n}{k} p_L^{n-k}\, p_H^k, \qquad k = 0, \ldots, n. \tag{III.20}$$

By construction, it holds that $\sum_{i=1}^n \mathbb{1}(Y_i = H) + \sum_{i=1}^n \mathbb{1}(Y_i = L) = n$. For the conditional distribution of S_n given that $C_n = k$, we obtain

$$S_n | C_n = k \sim \mathcal{N}((n-k)\, \mu_L + k\, \mu_H, (n-k)\, \sigma_L^2 + k\, \sigma_H^2). \tag{III.21}$$

We conclude

$$\mathbb{P}[S_n \leq x] = \sum_{k=0}^n p_{n,k}\, \Phi(x, \mu_{n,k}, \sigma_{n,k}^2), \tag{III.22}$$

where

$$\mu_{n,k} = (n-k) \cdot \mu_L + k \cdot \mu_H, \tag{III.23}$$

$$\sigma_{n,k}^2 = (n-k) \cdot \sigma_L^2 + k \cdot \sigma_H^2, \tag{III.24}$$

$$p_{n,k} = \binom{n}{k} \cdot p_L^{n-k} \cdot p_H^k \tag{III.25}$$

for $k = 0, \ldots, n$. From the behavior of mixtures under linear transformations, it then follows that

$$\mathbb{P}\Big[\bar{S}_n \leq x\Big] = \sum_{k=0}^n \bar{p}_{n,k}\, \Phi(x; \bar{\mu}_{n,k}, \bar{\sigma}_{n,k}^2) \tag{III.26}$$

with $\bar{\mu}_{n,k} = \frac{1}{n}\mu_{n,k}$, $\bar{\sigma}^2_{n,k} = \frac{1}{n^2}\sigma^2_{n,k}$ and $\bar{p}_{n,k} = p_{n,k}$ for $k = 0, \ldots, n$. Using this result, the VaR of $\bar{S}_n$ can be determined numerically by solving

$$\sum_{k=0}^{n} \bar{p}_{n,k}\, \Phi(\mathrm{VaR}_\alpha\left[\bar{S}_n\right], \bar{\mu}_{n,k}, \bar{\sigma}^2_{n,k}) = 1 - \alpha. \tag{III.27}$$

From equation (III.3) and the representation of the partial moment generating for a normal mixture given in equation (II.10), we conclude

$$\begin{aligned}
\mathbb{M}_{\bar{L}^v_n}(\gamma_i) &= \exp\Big(-\gamma_i\, \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\Big)\ \mathbb{E}\Big[\exp\Big(\gamma_i\, \bar{S}_n\Big)\ \mathbb{1}(\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right])\Big] + (1-\alpha) && \text{(III.28)}\\
&= \sum_{k=0}^{K} \bar{p}_{k,n}\ \exp\left(\gamma_i\ \left(\bar{\mu}_{k,n} - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right) + \frac{1}{2}\gamma_i^2 \bar{\sigma}_k^2\right) \\
&\qquad \left(1 - \Phi\left(\frac{\mathrm{VaR}_\alpha\left[\bar{S}_n\right] - \bar{\mu}_{k,n}}{\bar{\sigma}_{k,n}} - \gamma_i\, \bar{\sigma}_{k,n}\right)\right) + (1-\alpha). && \text{(III.29)}
\end{aligned}$$

Details Example 3 iii): Under a mixture assumption with perfectly dependent state indicators, the distribution of S_n is a two-state mixture with

$$\mathbb{P}[S_n \le x] = p_L \Phi\Big(x; n\mu_L, n\,\sigma_L^2\Big) + p_H \Phi\Big(x; n\,\mu_H; n\,\sigma_H^2\Big)\,. \tag{III.30}$$

Therefore, we can use the same arguments as in part ii) of Example 3 to compute the VaR and the moment generating function of $\bar{L}^v_n$. Furthermore, the unconditional covariance of the risks in the pool can be calculated as follows

$$\begin{aligned}
\mathrm{cov}\,[X_1, X_2] &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\ \mathbb{E}[X_2] && \text{(III.31)}\\
&= \pi_L \mathbb{E}[X_1\, X_2 \mid Y = L] + \pi_H \mathbb{E}[X_1\, X_2 \mid Y = H] - (\mathbb{E}[X_1])^2 && \text{(III.32)}\\
&= \pi_L\, \mathbb{E}[X_1 \mid Y = L]\ \mathbb{E}[X_2 \mid Y = L] \\
&\qquad + \pi_H\, \mathbb{E}[X_1 \mid Y = H]\ \mathbb{E}[X_2 \mid Y = H] - \mathbb{E}[X_1]^2 && \text{(III.33)}\\
&= \pi_L\, \mu_L^2 + \pi_H\, \mu_H^2 - (\pi_L\, \mu_L + \pi_H\, \mu_H)^2. && \text{(III.34)}
\end{aligned}$$

Details Example 4: Suppose that $\bar{S}_n$ follows a K-component normal mixture with the state-specific parameters μ_k and σ_k and a state indicator Y with $\mathbb{P}[Y = k] = p_k$ for $k = 1, \ldots, K$. Then,

it holds that

$$\mathrm{AVaR}_\alpha\Big[\bar{S}_n\Big] = \mathbb{E}\Big[\bar{S}_n \mid \bar{S}_n > \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big]\Big] \tag{III.35}$$

$$= \frac{1}{\alpha}\sum_{k=1}^{K} p_k\,\mathbb{E}\Big[\bar{S}_n \cdot \mathbb{1}(\bar{S}_n > \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big]) \mid Y = k\Big]\,. \tag{III.36}$$

The conditional expectations in the last expression correspond to partial expectations for normally distributed random variables. Using the results of Winkler et al. (1972, p. 292) and $\mathbb{E}[Z\,\mathbb{1}(Z < a)] + \mathbb{E}[Z\,\mathbb{1}(Z \geq a)] = \mathbb{E}[Z]$, it follows that

$$\mathbb{E}\Big[\bar{S}_n \cdot \mathbb{1}(\bar{S}_n > \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big]) \mid Y = k\Big] = \mu_k - (-\sigma_k\,\varphi(v_k) + \mu_k\,\Phi(v_k)) \tag{III.37}$$

$$= \sigma_k\,\varphi(v_k) + \mu_k\,\Phi(-v_k) \tag{III.38}$$

with $v_k = \frac{\mathrm{VaR}_\alpha[\bar{S}_n] - \mu_k}{\sigma_k}$.

IV Additional Example

We finally provide an additional example using a simple discrete loss distribution. In particular, we assume that the losses $(X_1, \ldots, X_n)$ are independent and identically distributed with

$$\mathbb{P}[X_i = 0] = p_L \qquad \text{and} \qquad \mathbb{P}[X_i = h] = p_H \tag{IV.1}$$

for $i = 1, \ldots, n$. Then, we can rewrite X_i as $X_i = h\,Z_i$ with a Bernoulli random variable Z_i, i.e., $\mathbb{P}[Z_i = 0] = p_L$ and $\mathbb{P}[Z_i = 1] = p_H$. Given these assumptions, $B_n := \sum_{i=1}^n Z_i$ has a Binomial distribution and it holds that

$$\bar{S}_n = \frac{k\,h}{n} \qquad \Leftrightarrow \qquad B_n = k \tag{IV.2}$$

for $k = 0, \ldots, n$, which implies

$$\mathbb{P}\Big[\bar{S}_n = \frac{h\,k}{n}\Big] = \mathbb{P}[B_n = k] = \binom{n}{k} \cdot p_L^{n-k} \cdot p_H^k. \tag{IV.3}$$

Assuming an exponential utility function with the risk aversion parameter γ_i, we obtain

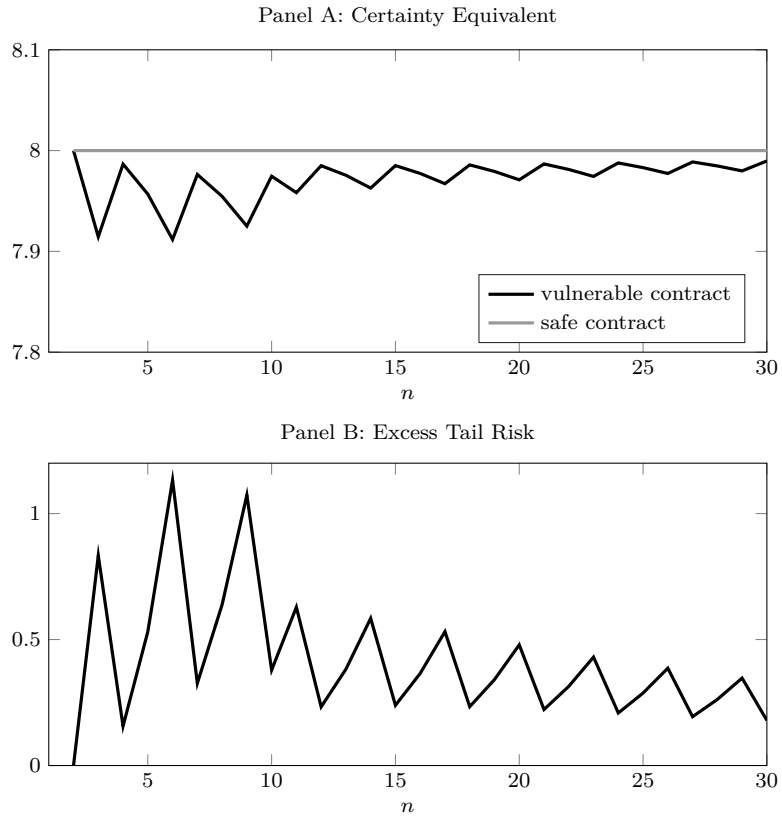
$$\mathbb{E}[u_i(W_{i,n})] = 1 - \sum_{k=0}^{n} \mathbb{P}\left[\bar{S}_n = \frac{h\,k}{n}\right] \exp\left(-\gamma_i \left(w_{0,i} - \pi - \max\left\{\frac{h\,k}{n} - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\,;\,0\right\}\right)\right) \tag{IV.4}$$

for the expected utility from a buying a vulnerable insurance contract from a company with a total risk pool of size n. To determine the VaR_α of $\bar{S}_n$, we simply use the general VaR definition for the probability distribution given in (IV.3) and the corresponding AVaR_α can be computed as follows

$$\begin{aligned}\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \frac{1}{\alpha}\Big\{&\mathbb{E}\left[\bar{S}_n \mathbb{1}\left(\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right)\right] \\ &+ \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\left(\alpha - \mathbb{P}\left[\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right]\right)\Big\}.\end{aligned} \tag{IV.5}$$

In this case, the relationship between the pool size and the expected utility depends on the distributional parameters. We illustrate the expected utility and the excess tail risk as a function of n for $h = 8$, $p_L = \frac{3}{4}$ (and thus $p_H = \frac{1}{4}$), $\pi = 2$, $w_{0,i} = 10$, $\gamma_i = 0.5$ and $\alpha = 5\%$ in Figure IV.1. With these parameters, the results are qualitatively similar to the setting with independent mixtures of normals that we discussed as Example 3 ii). Note, however, that the change in the policyholders' utility level is partly associated with a varying default probability, which is caused by the discreteness of the loss distribution.

Figure IV.1: Additional Example – Binomial Distribution



Note: This figure illustrates the case of a VaR-based equity capital under the assumption of independent and identically distributed losses with $\mathbb{P}[X_i = 0] = 0.75$ and $\mathbb{P}[X_i = 8] = 0.25$. Furthermore, we assume an exponential utility function with $\gamma_i = 0.5$ and suppose that $w_{0,i} = 10$, $\pi = 2$ and $\alpha = 0.05$. Panel A depicts the certainty equivalents of buying the safe insurance contract (gray line) and the vulnerable insurance contract (black line) as a function of the pool size n. Panel B presents the excess tail risk $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ as a function of n.

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