Option-Implied Solvency Capital Requirements

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Abstract: We propose a new methodology for measuring long-term market tail risk. By incorporating information from option markets, we obtain risk estimates that quickly react to new information and we circumvent the paucity of time-series data at long horizons. We demonstrate the implementation of our approach for one-year interest rate risk and equity risk within the Solvency II framework. On average, our risk estimates for interest rate changes are larger than the shocks according to the current Solvency II standard formula and there are pronounced differences in the dynamics of these risk estimates. For equity indices, our results reveal a substantial time variation in the long-term tail risk perceived by market participants. Using our option-implied estimates in an internal model for market risk shows that the documented differences to the standard formula are economically significant. Overall, our methodology can offer additional insights for market risk regulation and for internal risk management practices.

Keywords: Capital Regulation, Solvency II, Options, Risk Management, Value-at-Risk

JEL classifications: G12, G22, G32, G38

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1 Introduction

Tail risk assessments over long horizons, such as one year, are an integral part of risk management techniques in insurance companies and pension funds. In addition, they are increasingly adopted for insurance regulation. Both major regulatory frameworks in Europe, Solvency II and the Swiss Solvency Test (SST), rely on tail risk measures with a one-year horizon for setting capital requirements. The required risk estimation is particularly challenging for market risk because only limited amounts of data are available at a yearly horizon and the distribution of the relevant risk factors often varies substantially over time.

Solvency II, which we will mainly focus on, uses the Value-at-Risk (VaR) at the 99.5% probability level over a one-year horizon for setting solvency capital requirements, so that the available capital can absorb potential losses in 199 out of 200 years.[1] The European Insurance and Occupational Pension Authority (EIOPA) estimates corresponding shocks from time series of *highly overlapping* one-year risk factor changes (CEIOPS, 2010). This approach has not only been challenged due to statistical problems arising from the large autocorrelation of overlapping data,[2] it has also been criticized more fundamentally due to its backward-looking nature. For example, Eling and Pankoke (2014) argue that

> "[...] research should investigate whether it is possible to calibrate capital requirements using factors other than historical data in order to mitigate backward-looking characteristics. Insurers should not only be ready for the last, but also for the next crisis."

In this paper, we develop a new methodology for the measurement of long-term market tail risk which reduces the reliance on past price data by incorporating information from options on the relevant risk factors. More specifically, we propose a two-step procedure for the calculation of long-term tail risk estimates that builds on ideas from the literature on extracting option-implied physical distributions (see, e.g., Bliss and Panigirtzoglou, 2004, Liu et al., 2007 or Ghysels and Wang (2014)) and, in particular, on a technique recently proposed by Huggenberger et al. (2018): First, we extract the relevant risk-neutral distributions from option prices. Second, we apply a parametric specification of the (projected) stochastic discount factor (SDF) for transforming risk-neutral into the corresponding physical probabilities. To obtain estimates for the parameters of the

[1]Details about Solvency II can be found in the Delegated Regulation EIOPA (2014).

[2]Mittnik (2016) concludes that "an implementation of the standard formula with the currently proposed calibration settings for equity–risk is likely to produce inaccurate, biased and, over time, highly erratic capital requirements".

projected SDF, we exploit information from the gap between the variances under the risk-neutral pricing measure and the physical probability measure. The tails of the resulting distributions are then used to calculate risk forecasts and capital requirements.

We implement this methodology for interest rate and equity risk. Our analysis of long-term interest rate tail risk relies on swaptions with one year to maturity. On each forecasting day, we recover the distribution of swap rate changes under the forward risk-neutral measure for selected tenors from prices of receiver and payer swaptions across a range of strikes.[3] Our approach exploits a special feature of EUR-swaptions, whose payoffs on the expiration date only depend on a single swap rate, so that we can recover information on different rates separately without additional assumptions on the term structure on that day. For long-term equity tail risk, we propose to use the prices of index calls and puts with maturities close to one year and again different strikes. In both cases, we assume a mixture of normals for recovering the risk-neutral distribution.[4] Although the mixture structure substantially extends the flexibility compared to simple Gaussian models, it allows for straightforward extensions of standard pricing techniques and a closed-form solution for the measure change, so that we can derive analytical characterizations of long-term equity and interest rate risk forecasts.

In our empirical analysis, we investigate tail risk estimates for 5-, 10- and 20-year EUR interest rates as well as one-year tail risk forecasts for the Eurostoxx 50, the S&P 500 and the FTSE 100. For each of these risk factors, we compute option-implied risk-neutral and physical one-year 99.5%-VaR-forecasts at the end of each month between 01/2006 and 12/2019. We compare our results to the shocks according to the current Solvency II standard formula and, for interest rate risk, to a revised proposal that EIOPA recently issued in the context of its "2020 Review of Solvency II" (EIOPA, 2020).[5] Furthermore, we include the following standard benchmarks: simple quantile estimates derived from a normal distribution with monthly data and time scaling as well as simulated tail risk forecasts that are derived from (E)GARCH models estimated with daily data. The Gaussian benchmark VaRs can be seen as an implementation of the econometric approach proposed by the

[3] For the conversion between spot and swap rates, we rely on the assumption of a small and deterministic spread. Note that the Solvency II basic risk-free term structure typically also relies on swap rates (European Commission, 2014, Articles 44 and 45).

[4] Note that standard exponential specifications of the pricing kernel restrict the application of alternative distributions with Pareto tails for modeling the risk-neutral probability law.

[5] See https://www.eiopa.europa.eu/content/opinion-2020-review-of-solvency-ii for this review process.

Swiss Solvency Test (Finma, 2017).[6]

Over our sample period, the average option-implied tail risk estimate for an increase in the 10-year rate corresponds to 2.01 percentage points and the corresponding estimate for an interest rate decrease is given by 1.39 percentage points.[7] The implied distributions of interest rate changes vary substantially over time and often exhibit pronounced asymmetries, which seem to reflect expected changes in the economic environment such as the reduction of interest rates during the financial crisis. The option-implied downward (upward) shocks for the 10-year rate range between 0.97 and 2.97 (0.63 and 2.84) percentage points. Our downward interest rate shocks extracted from option prices are always larger than the corresponding Solvency II shocks with the average implied shock being almost twice as large as the average shock according to the current Solvency II standard formula.[8] For the upward shocks, we observe higher levels of the Solvency II benchmark during the first years of our sample whereas the implied tail risk estimates are substantially higher during the second half of our sample period. We also document pronounced differences between option-implied risk estimates and selected benchmark forecasts relying on past interest rate changes. In particular, during the current low interest rate environment, option-implied upward (downward) shocks are higher (lower) than the corresponding risk forecasts calibrated from past data.

In our equity risk analysis, we document average option-implied one-year 99.5%-VaRs of 47.76% for the Eurostoxx 50, 45.52% for the S&P 500 and 42.97% for FTSE 100. For all markets, we find the lowest tail risk estimates in 2006 (Eurostoxx: 38.41%) and the highest estimates during the financial crisis (Eurostoxx: 62.33%). The average level of the option-implied tail risk estimates is in each case larger than the unadjusted shock of 39% according to the Solvency II standard formula. For the Eurostoxx 50, the average difference amounts to 8.76% and it can reach values of more than 20% over time. Applying the macroprudential "symmetric" adjustment to the option-implied tail risk estimates partially off-sets their time-series variation. The standard deviation of the adjusted option-implied estimates is lower than the standard deviation of the adjusted Solvency II shocks. Comparing the option-implied equity risk estimates to forecasts based on past data, we find that VaR-forecasts derived from a normal distribution and EGARCH models are lower during crisis

[6]Note, however, that the SST relies on the Expected Shortfall instead of the VaR.

[7]Our results for the different maturities are qualitatively similar.

[8]Interestingly, a similar average difference is also observed between the shocks according to the current and the revised Solvency II interest rate methodology. Over our sample period, the average downward shock for the revised approach corresponds to 1.50 percentage points.

periods. In contrast, risk forecasts generated with Filtered Historical Simulation typically exceed our option-implied estimates.

Since the change from the risk-neutral to the physical probability measure is of particular relevance for our methodology, we test the robustness of our results with respect to different implementations of our moment-based SDF calibration.[9] In particular, we consider alternative benchmark forecasts for the physical variance and we vary the window as well as the objective function for the moment matching. In addition, we consider different state numbers for the normal mixture models. Overall, our results for interest rate and equity risk are relatively stable under these variations.

Finally, we investigate the economic importance of the documented differences by studying the impact of using option-implied shocks within a partial internal model for two stylized life insurance companies. We compare the overall market risk capital requirements from this internal model with the requirements according to the standard formula and the Gaussian benchmark based on monthly risk factor changes as proposed by the SST. We find that the standard formula in its current form typically implies lower capital requirements than the benchmarks. Furthermore, option-implied estimates indicate a higher level of risk during the financial crisis as well as during the recent low interest rate period compared to the Solvency II standard formula.

Our research contributes to the literature that analyzes market risk capital requirements for insurance companies in the context of Solvency II or the Swiss Solvency Test (Gatzert and Martin, 2012; Braun et al., 2014; Eder et al., 2014; Braun et al., 2017; Laas and Siegel, 2017). To the best of our knowledge, option-implied methods have not been considered in this context yet, although they seem to be well suited for at least two reasons: First, they overcome the drawbacks of conventional statistical methods based on historical data when working with long holding periods and, second, they quickly react to new information and changing market conditions.

With respect to Solvency II, we furthermore contribute to the ongoing review process, where EIOPA (2018a) identified a "severe under-estimation of the risks" with the current interest rate shocks. Our results tend to support the new EIOPA (2020)-proposal. Whereas the current Solvency II shocks are often significantly smaller than the option-implied tail risk estimates, the revised Solvency II shocks are typically closer to these benchmarks – particularly during the current low

[9] In line with the existence of large risk premia in the tails, we find pronounced differences between our risk-neutral and physical VaR-forecasts.

interest rate environment.

Beyond the determination of capital requirements, we contribute to the large body of literature analyzing the advantages of using forward-looking information contained in option prices.[10] Option-implied Value-at-Risk forecasts have recently been studied by Ghysels and Wang (2014), Barone Adesi (2016) and Huggenberger et al. (2018).[11] Whereas those papers focus on short-maturity options (typically around 30 days or less), we focus on options with long maturities, which allows us to extract one-year risk forecasts. Option-implied interest rate tail risk has received considerably less attention so far, but several studies extract probability distributions from interest rate derivatives. For example, Trolle and Schwartz (2014) and Hattori et al. (2016) derive swap rate distributions from swaption quotes, Li and Zhao (2009) and Ivanova and Puigvert Gutiérrez (2014) recover state-price densities from interest rate caps and options on EURIBOR futures, respectively, and Beber and Brandt (2006) compare probability densities implied by options on U.S. Treasury bond futures before and after macroeconomic announcements.

Finally, the present work is related to the econometric literature on measuring long-term equity risk (Guidolin and Timmermann, 2006; Engle, 2011) and long-term interest rate risk (Engle et al., 2017) with methods that rely on historical data. Those papers apply parametric time-series techniques to extrapolate risk forecasts from daily or monthly data to longer time horizons. We propose an option-implied alternative to these techniques and compare our results to a selection of time-series benchmarks.[12]

The structure of this paper is as follows. Section 2 provides the necessary background on the Solvency II standard formula. In Section 3, we describe our methodology for extracting option-implied long-term VaR estimates. Our empirical results on one-year interest rate and equity tail risk are summarized in the Sections 4 and 5. Section 6 introduces a partial internal model based on option-implied shocks and compares the resulting capital requirements for market risk to the Solvency II standard formula. Section 7 concludes. The Appendices provide some additional information on Solvency II and swaptions as well as additional empirical results.

[10] For equity options, a comprehensive literature overview is provided by Figlewski (2018) and an extensive review on methods for estimating forward-looking interest rate distributions with a focus on short-term rates is provided by Wright (2017).

[11] A related strand of literature (see, for example, Bollerslev and Todorov, 2011 or Andersen et al., 2017) studies the extraction of left tail jump risk from derivatives and investigates implications for the equity risk premium.

[12] Note that our approach partially relies on results from this econometric literature as we use GARCH-based *volatility*-forecasts to calibrate the pricing kernel.

2 Background on Solvency II

Under Solvency II, the overall solvency capital requirement (SCR) is determined as the Value-at-Risk (VaR) at the confidence level of 99.5% for the one-year loss in basic own funds. Basic own funds BOF_t at time t are defined as the difference between the values of assets A_t and liabilities L_t (EIOPA, 2014), that is,

$$BOF_t = A_t - L_t. \tag{1}$$

Furthermore, the VaR of a random loss L at the confidence level β can be defined as the lower β-quantile of L, i.e.

$$\mathrm{VaR}_\beta[L] := \inf\{x \in \mathbb{R} \mid \mathbb{P}[L \leq x] \geq \beta\}. \tag{2}$$

It thus corresponds to the lowest loss level which is not exceeded with a probability of at least β.[13] Denoting the change in basic own funds by $\Delta BOF = B_{t+1} - B_t$, we formally obtain

$$SCR = \mathrm{VaR}_{99.5\%}[-\Delta BOF] \tag{3}$$

for the overall solvency capital requirements.

Insurance companies can either use (partial) internal models, which need to be approved by the regulator, or they can apply the standard formula in order to calculate the SCR. Most of the companies currently apply the standard formula.[14] The standard formula is a bottom-up approach: It aggregates the risk from n sub-modules into the overall (basic) SCR using the following square-root aggregation rule

$$SCR = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j} \cdot SCR_i \cdot SCR_j}, \tag{4}$$

where $\rho_{i,j}$ is a predefined correlation parameter for the module pair (i, j), $i, j = 1, \ldots, n$, and SCR_i is the capital requirement for (sub-)module i, $i = 1, \ldots, n$. The main risk modules are market risk,

[13] For continuous distributions, the VaR satisfies the simpler condition $\mathbb{P}[L \leq \mathrm{VaR}_\beta[L]] = \beta$.

[14] See e.g. Milliman (2019).

underwriting risk (life, non-life, health), counterparty default risk and the risk related to intangible assets. For each main risk module, there are mostly further sub-modules. Importantly, the 99.5%-VaR objective is applied to determine the capital requirement for each (sub-)module individually (EIOPA, 2014, p. 7).[15]

We focus on capital requirements for interest rate risk SCR_{int} and equity risk SCR_{eq}, which are part of the market risk sub-module.[16] BaFin (2011) reports that interest rate risk is the most important risk category for life insurance companies and equity risk is the most important risk type for non-life insurers within the market risk sub-module. According to the standard formula, the SCR for equity and interest rate risk are given by the loss in basic own funds resulting from instantaneous shocks to the price of equity investments and to the term structure. The SCR of module i is thus formally calculated based on

$$SCR_i = -\Delta BOF \,|\, shock_i, \tag{5}$$

where $shock_i$ indicates the shock scenario for the relevant risk factors of module i.

For equity risk, CEIOPS (2010) calibrates the corresponding shock with a non-parametric 99.5%-VaR estimator applied to *overlapping* one-year returns that are calculated from daily data for *each* day of the sample period. From a statistical point of view, this calibration is problematic due to a large overlap in the yearly returns that are used to calculate empirical quantiles (Mittnik, 2016). A symmetric adjustment is added to this baseline shock which works in a counter-cyclical manner and has to be seen as a macroprudential tool. The precise rules for calculating the total shock according to the standard formula and more details on the symmetric adjustment are provided in Appendix A.

According to Article 165 of the Delegated Regulation, the capital requirement for interest rate risk is based on upward and downward shock scenarios for the term structure of interest rates. More specifically, the maximum loss in basic under funds under both scenarios determines the interest rate risk SCR. The calibration of the corresponding interest rate shocks is again based

[15] However, this bottom-up procedure does not necessarily ensure that the 99.5%-VaR objective is also met on the company level (Pfeifer and Strassburger, 2008). EIOPA (2014) tries to address this issue by choosing appropriate correlations for the square root aggregation (4).

[16] The further market risk sub-modules, which are not the focus of the present work, are property risk, spread risk, market concentration risk and currency risk. Note that, overall, market risk is the most important risk category for life and non-life insurance companies according to EIOPA (2018b).

on *overlapping* one-year changes in the spot rates that are calculated on a daily basis (CEIOPS, 2010; EIOPA, 2016). For each maturity, the upward and downward shocks are calibrated to reflect the 99.5%-quantile and the 0.5%-quantile of the relative one-year change in the spot rate (EIOPA, 2016, p. 65).[17] This procedure implicitly assumes perfectly dependent shocks to the term structure. The specific calculation of the Solvency II interest rate shocks is explained in Appendix A, where we also introduce the recently proposed adjusted shocks (EIOPA, 2020).

To sum up, the equity shock and the maturity-specific shocks to the term structure of interest rates calculated to reflect 0.5%- or 99.5%-quantiles of one-year risk factor changes are key inputs for the market risk module in Solvency II.

3 Methodology

In this section, we first present a general methodology to determine distributions of long-term risk factor changes based on option prices. Then, we discuss the specific implementation of this methodology for long-term interest rate risk and equity risk.

3.1 Physical Option-Implied Distributions

Our approach builds on ideas developed by Bliss and Panigirtzoglou (2004), Liu et al. (2007) and Ghysels and Wang (2014), who extract implied physical distributions from option prices by combining risk-neutral distribution forecasts with a parametric form of the Stochastic Discount Factor (SDF) that connects the risk-neutral and the physical probability law. This idea has recently been exploited for short-term equity tail risk measurement by Huggenberger et al. (2018). We adapt this methodology to long-term risk measurement for equity and interest rate risk factors.

The proposed methodology relies on the time-t prices of derivatives with maturity at time $t+\tau$ to extract information about the distribution of the risk factor changes $X_{t,\tau}$ over the period $[t, t+\tau]$. More specifically, we assume that we observe the market prices $p_{t,1}, \dots, p_{t,M}$ of M derivatives, whose payoffs at maturity only depend on $X_{t,\tau}$. In this case, the payoffs at maturity can be written as $P_{t+\tau,i} = v_i(X_{t,\tau})$ with measurable functions $v_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, M$.

[17] EIOPA does not directly use the empirical quantiles of the relative spot rate changes even if they argue that this would also be a reasonable approach (EIOPA, 2016, p. 65). Instead, a technique relying on principal component analysis is interposed to represent the relative spot rate changes as linear combinations of weighted principal scores for each observation date (CEIOPS, 2010; EIOPA, 2016).

Under no-arbitrage, the price of these derivatives can be represented using the well-known risk-neutral pricing equation

$$p_{t,i} = B_{t,\tau}\, \tilde{\mathbb{E}}_t[v_i(X_{t,\tau})], \tag{6}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to a forward martingale measure $\tilde{\mathbb{P}}$ given the information at time t and $B_{t,\tau}$ denotes the time-t price of a zero-coupon bond with a face value of one and time to maturity τ.[18]

Furthermore, we exploit the existence of a stochastic discount factor (SDF) $M_{t,\tau}$ in arbitrage-free markets (Hansen and Richard, 1987). More specifically, we can rely on the *projected* SDF $M^*_{t,\tau} := \mathbb{E}_t[M_{t,\tau} \,|\, X_{t,\tau}]$ for the pricing of payoffs only depending on the risk-factor change $X_{t,\tau}$ (Rosenberg and Engle, 2002). Using this projection, the time-t price of the payoff $P_{t+\tau,i} = v_i(X_{t,\tau})$ can be written as expectation $\mathbb{E}_t$ under the physical measure, i.e.,

$$p_{t,i} = \mathbb{E}_t[M^*_{t,\tau}\, v_i(X_{t,\tau})]. \tag{7}$$

By the definition of the conditional expectation, there is a measurable function $m^*_{t,\tau} : \mathbb{R} \to \mathbb{R}$ that satisfies $M^*_{t,\tau} = m^*_{t,\tau}(X_{t,\tau})$. If we assume that the distribution of $X_{t,\tau}$ is absolutely continuous, then the comparison of equations (6) and (7) implies

$$f_{t,\tau}(x) = \frac{\tilde{f}_{t,\tau}(x)/m^*_{t,\tau}(x)}{\int \tilde{f}_{t,\tau}(s)/m^*_{t,\tau}(s)\, ds}, \tag{8}$$

where $\tilde{f}_{t,\tau}$ denotes the density with respect to the forward risk-neutral measure[19] $\tilde{\mathbb{P}}$ and $f_{t,\tau}$ is the physical density of $X_{t,\tau}$.[20] Similar results are e.g. used by Bliss and Panigirtzoglou (2004), Liu et al. (2007) as well as Huggenberger et al. (2018) for equity index returns and by Li and Zhao (2009) for interest rate distributions. To implement this general relationship for the extraction of long-term tail risk, we add the following two approximation arguments.

[18] In contrast to Huggenberger et al. (2018), we use the forward risk-neutral measure instead of the standard risk-neutral measure, which allows us to avoid the assumption of a deterministic discount rate for the option payoff in our interest rate application. The forward risk-neutral measure is obtained by choosing a zero-coupon bond with maturity at $t + \tau$ as numeraire. Cf., e.g., Li and Zhao (2009) or Wright (2017).

[19] In the following, we will simply refer to $\tilde{f}_{t,\tau}$ as the "risk-neutral" density.

[20] See Section I in the Online Appendix for details.

First, we assume that the risk-neutral density $\tilde{f}_{t,\tau}$ can be approximated by a finite mixture of normals. In contrast to normal distributions, mixtures of normals are highly flexible and allow for skewness, excess kurtosis and multimodality. Besides, standard pricing techniques can often be extended from simple normal distributions to normal mixtures. Due to this flexibility and analytical tractability, they have been used in a number of previous studies to extract probability distributions implied from equity and interest rate derivatives (see, e.g., Melick and Thomas, 1997; Liu et al., 2007; Huggenberger et al., 2018). Under the mixture assumption, the risk-neutral density function of $X_{t,\tau}$ conditional on the information available at time t can be written as

$$\tilde{f}_{t,\tau}(x) = \sum_{k=1}^{K} \tilde{\pi}_{t,k}\, \varphi(x; \tau\, \tilde{m}_{t,k}, \tau\, \tilde{\sigma}^2_{t,k}), \tag{9}$$

where $\varphi(\,\cdot\,; m, \sigma^2)$ is the probability density of a normal distribution with mean m and variance σ^2. $\tilde{\pi}_{t,k} \geq 0$, $k = 1, \ldots, K$, are weights of the mixture components, which are often interpreted as state probabilities. They must satisfy $\sum_{k=1}^{K} \tilde{\pi}_{t,k} = 1$. Furthermore, $\tilde{m}_{t,k}$ and $\tilde{\sigma}_{t,k}$ are the annualized component-specific mean and standard deviation parameters.

Our second important approximation is a Taylor-series expansion for the logarithm of the unknown stochastic discount factor function $m^*_{t,\tau}$. In particular, we rely on the second-oder approximation

$$\ln m^*_{t,\tau}(x) = \beta_{t,\tau} + \gamma_t\, x + \delta_t\, x^2. \tag{10}$$

This specification obviously includes the standard exponential affine pricing kernel as a special case ($\delta_t = 0$), which follows from standard assumptions in asset pricing models and is known as "Esscher transform" in the actuarial literature (Gerber and Shiu, 1996).[21,22] Furthermore, it allows for U-shaped pricing kernels that have been found in empirical studies for equity and, in particular, for interest rate risk factors (Li and Zhao, 2009; Ivanova and Puigvert Gutiérrez, 2014).

Given the additional structure in equations (9) and (10), the measure change according to equation (8) boils down to a parameter transformation. It is straightforward to show that the option-implied physical density resulting from equation (8) is given by (Monfort and Pegoraro,

[21]See e.g. Rosenberg and Engle (2002) for exponentially affine specifications of the projected SDF. In this case, the asset-specific parameter γ_t is related to the agent's relative risk aversion.

[22]Accordingly, Monfort and Pegoraro (2012) refer to the specification in equation (10) as the "second order Esscher transform".

2012)

$$f_{t,\tau}(x) = \sum_{k=1}^{K} \pi_{t,k}\, \varphi(x; \tau\, m_{t,k}, \tau\, \sigma^2_{t,k}), \tag{11}$$

where

$$\tau\, m_{t,k} = \frac{\tau\, \tilde{m}_{t,k} + \tau\, \tilde{\sigma}^2_{t,k} \gamma_t}{1 - 2\, \delta_t\, \tau\, \tilde{\sigma}^2_{t,k}}\,, \qquad \tau \sigma^2_{t,k} = \frac{\tau\, \tilde{\sigma}^2_{t,k}}{1 - 2\, \delta_t\, \tau\, \tilde{\sigma}^2_{t,k}} \tag{12}$$

and

$$\pi_{t,k} = \frac{\tilde{\pi}_{t,k}\, W(\gamma_t,\, \delta_t,\, \tau \tilde{m}_{t,k}, \tau \tilde{\sigma}^2_{t,k})}{\sum_{j=1}^{K} \tilde{\pi}_{t,j}\, W(\gamma_t,\, \delta_t,\, \tau \tilde{m}_{t,j}, \tau \tilde{\sigma}^2_{t,j})}. \tag{13}$$

The function W is the so-called Second-Order Laplace Transform. For the normal distribution, it corresponds to[23]

$$W(\gamma,\, \delta, m,\, \sigma) = \frac{1}{\sqrt{1 - 2\, \delta\, \sigma^2}}\, \exp\left[\frac{1}{1 - 2\, \delta \sigma^2}\, \left(m\, \gamma + \frac{1}{2}\, \gamma^2\, \sigma^2 + m^2\, \delta\right)\right]. \tag{14}$$

According to (11)-(13), γ_t determines the magnitude of a risk premium that is applied to the state specific location parameters (similar to the adjustment under Black-Scholes assumptions) and δ_t determines a scaling factor that is applied to the mean and variance parameters.

For the calibration of the model parameters of the risk-neutral density $\tilde{\boldsymbol{\pi}}_t = (\tilde{\pi}_{t,1}, \ldots, \tilde{\pi}_{t,K})$, $\tilde{\boldsymbol{m}}_t = (\tilde{m}_{t,1}, \ldots, \tilde{m}_{t,K})$ and $\tilde{\boldsymbol{\sigma}}_t = (\tilde{\sigma}_{t,1}, \ldots, \tilde{\sigma}_{t,K})$ from equation (9) and the parameters of the discount factor function γ_t as well as δ_t from equation (10), we implement the following two-step approach for a given number of mixture components K.

We first maximize the fit between model prices $p^{mix}_{t,i}$ according to equation (6) and the corresponding market prices $p^{obs}_{t,i}$ for $i = 1, \ldots, m$. More specifically, we transform these prices into implied volatilities denoted by $iv(p^{mix}_{t,i})$ and $iv(p^{obs}_{t,i})$ and minimize the Mean Squared Error. Formally, we solve

$$\min_{\tilde{\boldsymbol{\pi}}_t, \tilde{\boldsymbol{m}}_t, \tilde{\boldsymbol{\sigma}}_t} \frac{1}{m} \sum_{i=1}^{m} \Big(iv(p^{obs}_{t,i}) - iv(p^{mix}_{t,i})\Big)^2 \tag{15}$$

[23]See Monfort and Pegoraro (2012, Appendix A). For $\delta = 0$, W simplifies to the well-known moment-generating function of the normal distribution.

under the restrictions $\tilde{\pi}_{t,k} \geq 0$ and $\sum_{k=1}^{K} \tilde{\pi}_{t,k} = 1$. In addition, we impose weak constraints on the range of the state-specific parameters to avoid problems in the numerical optimization.

Second, we determine the SDF parameters with a moment-based methodology that is based on the gap between the variance under the pricing measure $\tilde{\mathbb{P}}$ and the physical measure $\mathbb{P}$. In particular, we calibrate γ_t and δ_t by matching the option-implied physical variances derived from (9) to a series of benchmark forecasts based on past risk factor realizations.[24] In particular, we minimize the relative squared errors between these variances for a selection of forecasting dates $s = 1, \ldots, R$, while taking into account M option cross sections with time to maturity τ_k, $k = 1, \ldots, M$, at each date. Formally, we solve

$$\min_{\gamma_t, \delta_t} \; \frac{1}{R\,M} \sum_{s=1}^{R} \sum_{k=1}^{M} \left(\frac{\sigma^2_{s,mix}(\tau_k)}{\sigma^2_{s,bench}(\tau_k)} - 1 \right)^2, \tag{16}$$

where $\sigma^2_{s,mix}(\tau_k)$ and $\sigma^2_{s,bench}(\tau_k)$ denote the implied variance forecast and the benchmark variance forecast at time s for the time horizon τ_k. The choice of relative instead of absolute errors assigns similar weights to periods with different levels of volatility.[25] We implement this approach with a growing calibration window, i.e., we use all forecasting dates $s \leq t$ for the variance matching. In our baseline implementation, we choose EGARCH variance forecasts as benchmark estimates.[26]

Given the physical distribution parameters, we can easily compute option-implied quantiles (and thus the VaR) for the mixture distribution. In particular, the p-quantile of $X_{t,\tau}$ is obtained by solving

$$\sum_{k=1}^{K} \pi_{t,k} \, \Phi(Q_p[X_{t,\tau}];\, \tau m_{t,k}, \tau \sigma^2_{t,k}) = p, \tag{17}$$

where $\Phi(\,\cdot\,; m, \sigma^2)$ denotes the cumulative distribution function of a normal random variable with mean m and variance σ^2 (Huggenberger et al., 2018).

[24] The variance of normal mixture distribution is given by $\sigma^2_{t,mix}(\tau) = \sum_{k=1}^{K} \pi_{t,k} \left(m_{t,k}\tau - \sum_{k=1}^{K} \pi_{t,k} m_{t,k} \tau \right)^2 + \sum_{k=1}^{K} \pi_{t,k} \tau \sigma^2_{t,k}$.

[25] This is particularly relevant for our interest rate application due to substantial changes in the interest rate level over our sample period.

[26] Details on the implementation of our benchmark forecasts are provided in Section IV of the Online Appendix. For our baseline analysis, we use an EGARCH-specification with standard normally distributed innovations and a growing window to estimate the model parameters.

3.2 Implied Interest Rate Tail Risk

To apply the proposed methodology for the extraction of long-term interest rate risk, we rely on the prices of swaption contracts. In particular, we choose the change in the n-year swap rate between time t and $t+\tau$ as risk factor, i.e., we set $X_{t,\tau} = \Delta S^n_{t,\tau} := S^n_{t+\tau} - S^n_t$, where S^n_t is the swap rate of a contract with a maturity of n years starting at time t.[27] We propose to extract the distribution of $\Delta S^n_{t,\tau}$ from swaptions with expiry at $t+\tau$. Swaptions, which are typically traded in large and liquid over-the-counter markets, are among the most important interest rate derivatives (Trolle and Schwartz, 2014).

There are two basic types of swaption contracts: The holder of a *payer* swaption has the right (but not the obligation) to enter into an n-year payer swap at time $t+\tau$ with the fixed rate Y_S, which is referred to as the swaption strike. Similarly, the holder of a *receiver* swaption has the right to enter a receiver swap at the swaption maturity. To determine the payoffs of such contracts at time $t+\tau$, different conventions are used for so-called USD-swaptions and EUR-swaptions. We focus on EUR-swaptions for our analysis. The payoff of a EUR payer swaption with strike Y_S and yearly payments starting at time $t+\tau$ is given by

$$v(S^n_{t+\tau}) = (S^n_{t+\tau} - Y_S)^+ \sum_{i=1}^{n} \frac{1}{(1+S^n_{t+\tau})^i}, \tag{18}$$

and the payoff of the corresponding receiver swaption is equal to

$$v(S^n_{t+\tau}) = (Y_S - S^n_{t+\tau})^+ \sum_{i=1}^{n} \frac{1}{(1+S^n_{t+\tau})^i}. \tag{19}$$

The intuition behind these payoffs is that a flat yield curve with an interest rate equal to the swap rate $S^n_{t+\tau}$ is used for discounting.[28] An important implication for our analysis is that the EUR-swaption payoffs can be written as measurable functions of the single risk factor $X_{t,\tau} = \Delta S^n_{t,\tau}$ due to $S^n_{t+\tau} = S^n_t + \Delta S^n_{t,\tau}$. Accordingly, we can apply the methodology outlined in Section 3.1 to obtain

[27] Swap rates are the fixed rates at which market participants enter into a swap contract without making initial payments. These rates are typically close to spot rates derived from government bond yields for the same maturity. Interestingly, the so-called Solvency II "basic interest rate term structure" is based on swap rates rather than government bond yields whenever possible (European Commission, 2014, Article 44).

[28] In contrast, USD-swaptions use a selection of spot rates for discounting the swap payments. Some authors argue that the differences from these conventions are negligible (see Trolle and Schwartz (2014, p. 2316) and Brigo and Mercurio (2006, pp. 243)), so that our methodology can potentially be extended to USD-swaptions.

option-implied estimates for the physical distribution of swap rate changes over the period $[t, t+\tau]$ for different maturities n.

For this purpose, we impose the normal mixture assumption from equation (9) on $\Delta S^n_{t,\tau}$ and derive an approximate closed-form solution for swaption prices under this assumption in Appendix B.[29] Given a selection of market prices[30] for payer and receiver swaptions with different swaption strikes, we determine the risk-neutral mixture parameters of $\Delta S^n_{t,\tau}$ by solving the minimization problem described in equation (15). The transformation of prices into implied volatilities that we apply in this step is based on so-called "normal implied volatilities" following standard market conventions.[31] Given the estimates of the risk neutral distribution parameters, we apply the moment matching according to equation (16) to obtain estimates of the SDF parameters γ_t and δ_t.

With these results, we calculate quantile forecasts for swap rate changes from equation (17). Using the behavior of quantiles under monotonic deterministic transformations (Föllmer and Schied, 2011, p. 488), we also obtain the corresponding forecasts for the level of the swap rate at $t+\tau$. To make the results comparable to the *spot* rate shocks used in Solvency II, a deterministic spread between swap and spot rate can be considered[32], i.e., we assume that

$$R^n_{t+\tau} = S^n_{t+\tau} - c_t, \tag{20}$$

where $R^n_{t+\tau}$ is the n-year spot rate at time $t+\tau$ and c_t is a (small) deterministic but possibly time-dependent spread. Using this relationship and again exploiting the transformation behavior of quantiles, we obtain quantile approximations for the future spot rate and spot rate changes.

3.3 Implied Equity Tail Risk

To implement our methodology for the measurement of long-term equity risk, we choose $X_{t,\tau}$ as the logarithmic return of the relevant stock market index. Formally, we set $X_{t,\tau} = U_{t,\tau} =$

[29] Our tests indicate that the approximation seems to be sufficiently accurate (see Table B.1). We make use of this approximation formula because we are not aware of an exact analytical pricing formula for EUR-swaptions in the case of normal mixtures and we want to avoid Monte Carlo pricing, which would make the calibration computationally highly demanding.

[30] Details on our datasets are provided in Section 4.1.

[31] Details on the implementation of this conversion are provided in Online Appendix II.

[32] The approximation (20) is motivated by (i) the close relationship between swap and spot rates with the same maturity and (ii) by the Solvency II approach. We refer to the discussion in Appendix A.

$\log I_{t+\tau} - \log I_t$, where I_t denotes the level of a stock market index at time t.

With this choice, we can largely follow the methodology proposed by Huggenberger et al. (2018) and extract the risk-neutral index return distribution from the prices of standard index put and call options for a range of strike prices. Since we typically do not observe prices for options with exactly one year to maturity ($\tau = 1$), we select cross sections of index options whose maturities τ_1 and τ_2 are close to the one year horizon with $\tau_1 < 1$ and $\tau_2 > 1$.[33] We then apply the methodology outlined in Section 3.1 as follows.

For the calibration of the risk-neutral parameters, we use midquotes as market prices and compute model prices under the mixture assumption given by equation (9). This calibration is done for each cross-section separately. To implement the pricing in the equity case, we add the assumption of a constant risk-free rate over $[t, t+\tau]$, so that we can work with standard results for put and call prices (see, e.g., Melick and Thomas, 1997 and Huggenberger et al., 2018 for the mixture case).[34] Market and model prices are transformed into implied volatilities using the Black-Scholes model (Andersen et al., 2017). We solve the minimization problem given in equation (15) subject to the standard martingale constraint that can be rewritten as

$$\sum_{k=1}^{K} \tilde{\pi}_{t,k} \exp\left((-r_t^f + q_t + \tilde{m}_{t,k} + \frac{1}{2}\tilde{\sigma}_{t,k}^2)\,\tau \right) = 1, \tag{21}$$

where r_t^f and q_t are the annualized risk-free rate and the annualized dividend yield over $[t, t+\tau]$.

Following the previous literature on option-implied equity return distributions (Rosenberg and Engle, 2002; Bliss and Panigirtzoglou, 2004; Liu et al., 2007), we implement a linear specification of $\log m_{t,\tau}^*$ for the equity case.[35] We estimate the parameter γ_t by solving (16) based on the two cross-sections ($M = 2$), whereby the benchmark volatility estimates are obtained from daily index returns. Given these results, we can calculate the implied VaR for the maturities τ_1 and τ_2 using (17), which we then linearly interpolate to obtain an option-implied VaR-forecast at the one year horizon.

For a comparison with the Solvency II shocks, the implied VaR on the level of the logarithmic return is transformed to a percentage VaR. For this purpose, we can again exploit the behavior of

[33] A detailed description of the data and our filtering procedures will be provided in Section 5.1.

[34] Furthermore, the forward measure then corresponds to the standard risk neutral measure (Björk, 2009, p. 404).

[35] The more general quadratic specification of the log-SDF is considered in the robustness analysis.

quantiles under monotonic transformations. In particular, the discrete index return can be written as $R_{t,\tau} = \exp(U_{t,\tau}) - 1$, which implies

$$\mathrm{VaR}_\beta[-R_{t,\tau}] = Q_\beta[-R_{t,\tau}] = 1 - \exp(Q_{1-\beta}[U_{t,\tau}]). \tag{22}$$

4 Results for Interest Rate Risk

4.1 Data and Calibration

We obtain our swaption prices from Bloomberg and use the Bloomberg Volatility Cube (Bloomberg, 2018) to aggregate quotes from the available market sources and to convert these composite quotes consistently into normal implied volatilities even if they are originally quoted in terms of Black volatilities or prices.[36]

Market participants only provide quotes for at-the-money (ATM) and out-of-the money (OTM) swaptions. Thus, quotes for receiver swaptions are available for strikes smaller than the forward swap rate and quotes for payer swaptions are provided for strikes larger than the forward swap rate. The ATM volatility is quoted for a straddle position, that is, a portfolio of a receiver swaption and a payer swaption with the same characteristics and strike prices equal to the forward swap rate.[37]

We collect quotes for one-year into $\{5, 10, 20\}$-year contracts from 01/2006 until 12/2019 with a monthly frequency.[38] Since 2011, we obtain quotes for 11 strikes at each observation date. The strikes are given by the forward swap rate $\pm 200, \pm 150, \pm 100, \pm 50, \pm 25$ and ± 0 basis points.[39,40]

Based on these quotes, we calibrate two-state mixtures for each observation date by solving the minimization problem (15).[41] In order to convert model prices into implied normal volatilities, we

[36] We do not use the interpolation or extrapolation features provided by Bloomberg such that our results are purely based on actual market quotes. Our detailed settings for extracting the data with the Bloomberg Volatility Cube will be provided upon request.

[37] See e.g. Trolle and Schwartz (2014, p. 2313) for further details.

[38] Quotes for one-year swaption maturities and swap tenors of 5, 10 and 20 years, as well as many other combinations, are quoted on a daily basis. In contrast to equity options, we always get quotes for a *fixed* time to maturity, that is, exactly one year in our case.

[39] Before 2011, the quotes for ± 150 basis points are not provided.

[40] Analyzing reports by the TriOptima trade repository, Trolle and Schwartz (2014, Section 1.2) document that long-dated swaptions are actively traded. Furthermore, they find that ATM contracts tend to be most liquid and argue that OTM contracts also trade frequently since they are often used to hedge embedded options in callable bonds and structured products.

[41] Because of the relatively small number of prices, we cannot easily increase the number of states for our interest rate analyses.

follow the approach implemented by Bloomberg, that is, the conversion is based on the normal pricing formula presented in Appendix II.[42] To investigate whether the calibration of two-state models works reliably with eleven quotes, we run Monte-Carlo simulations, in which we generate option prices for the strikes that are typically available from a fully specified two-state model for the swap rate. The results presented in Section III suggest that the characteristics of the calibrated distributions are in most cases largely similar to the true distributions used for the price simulation.

Based on the calibrated two-state mixture models, we determine the time-varying and maturity-specific SDF parameters γ_t and δ_t by solving the minimization problem (16).[43] For determining the benchmark volatilities, daily swap rates from Bloomberg are used. Figure 1 shows the resulting forward risk-neutral and physical densities for the 10-year rate on two selected dates as well as the log ratios of these densities. It illustrates that the calibrated projected pricing kernels can exhibit U-shaped patterns in line with the results in Li and Zhao (2009) as well as Ivanova and Puigvert Gutiérrez (2014).

4.2 Option-Implied Estimates and Benchmarks

We first present option-implied quantile estimates for the level of the one-year ahead 5-, 10- and 20-year rates over time. The evolution of the corresponding 0.5%- and 99.5%-quantiles together with the level of the spot rates are shown in Figure 2. This figure documents a substantial time-variation in the quantiles that largely follows the level of the interest rates and it reveals pronounced asymmetries in the extracted interest rate distributions. During the first half of our sample, the gap between the current spot rate and the 0.5%-quantile of the future spot rate distribution is much wider than the gap between the current rate and the 99.5%-quantile. Differences are particularly pronounced during the financial crisis. This is reversed during the second half of the sample period when the gap between the current rate and the upper quantile of the future rate increases. Furthermore, the results for the three maturities are structurally similar. Therefore, we mainly focus on results for the 10-year rates in the following analysis.

Given the pronounced asymmetries, we separately investigate risk estimates for interest rate decreases and increases. In line with the 99.5%-VaR objective in Solvency II, we calculate implied

[42] In line with Bloomberg, we use zero bond prices which are based on EUR swap rates.

[43] We apply a growing estimation window starting with 84 observations, which corresponds to the first half of our sample. The SDF parameters applied for the first years of our data are thus in-samples estimates.

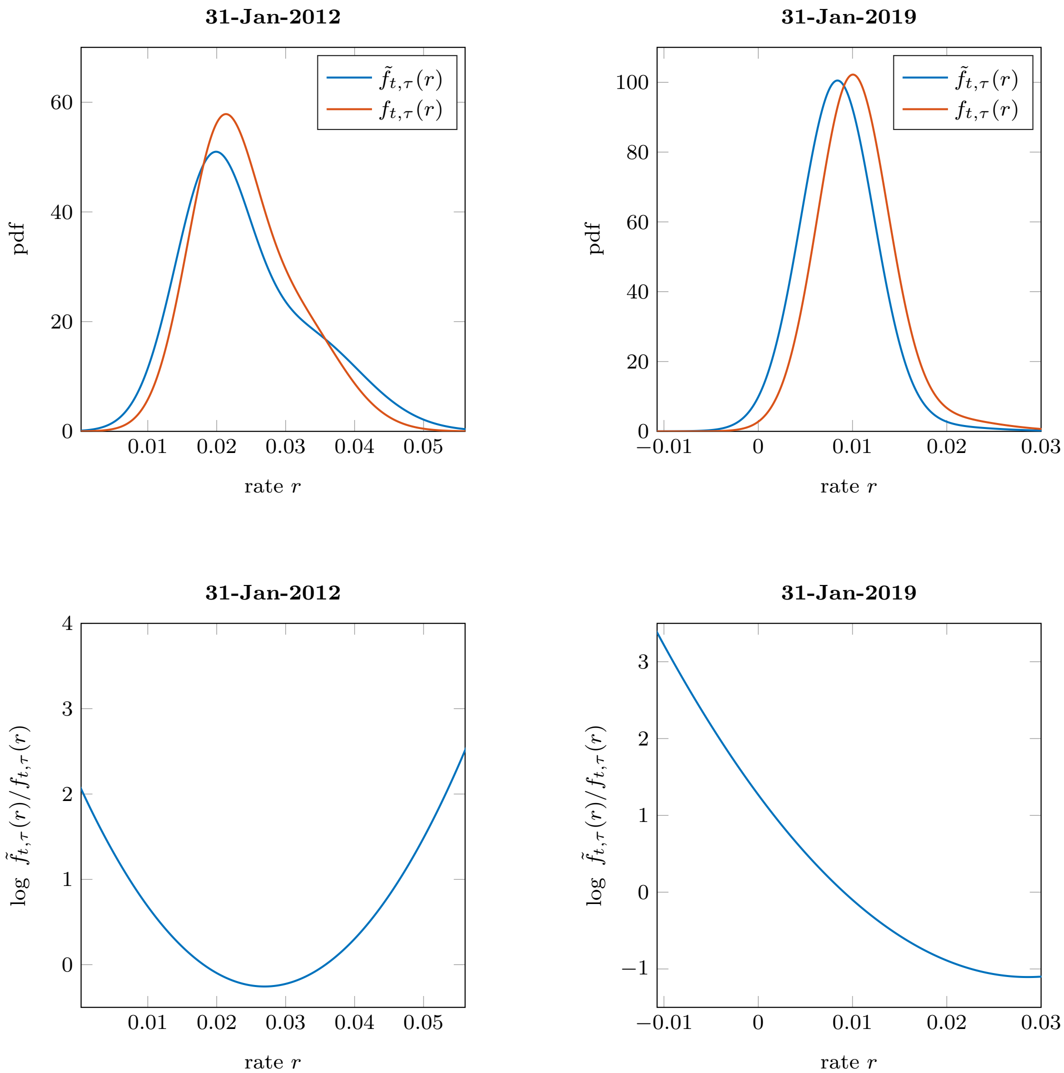
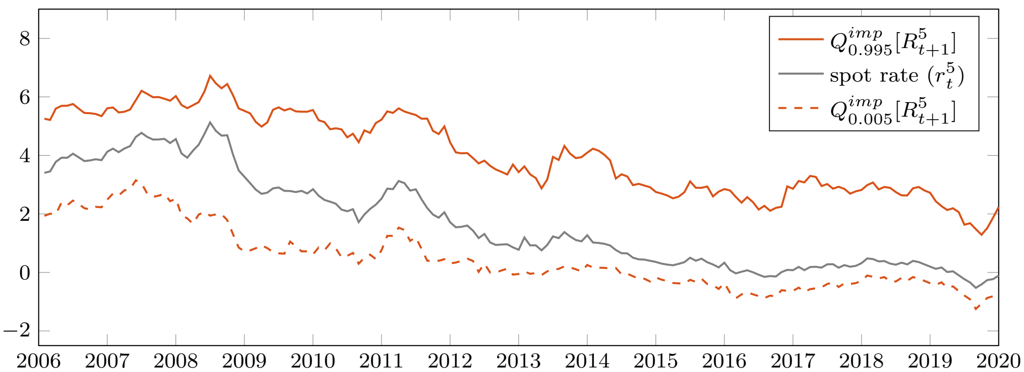
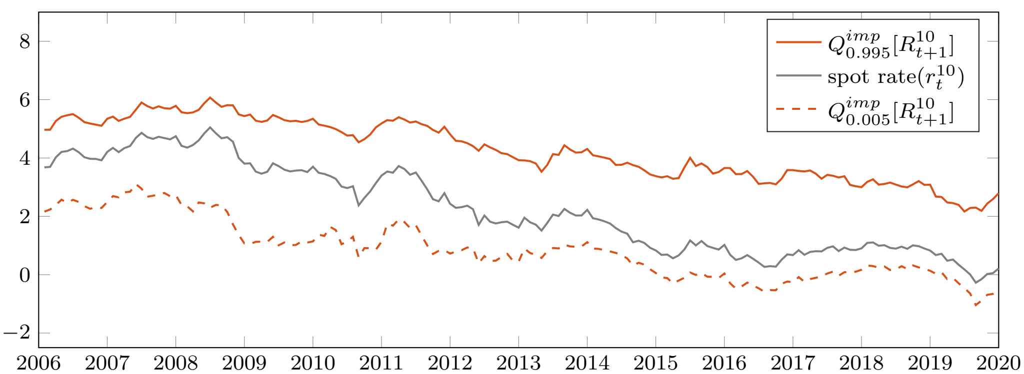


Figure 1: This figure shows the risk-neutral and the physical density functions of the one-year ahead 10-year rate as well the (log-)ratios of these densities on two selected forecasting days. The risk-neutral distributions are modeled as normal mixtures with two components, whose parameters are extracted from swaption prices. The log-SDF is approximated by a quadratic function and its parameters are calibrated by matching the physical one-year interest rate variance to EGARCH benchmark forecasts.

Panel A: Implied Quantiles for the 5-Year Rate



Panel B: Implied Quantiles for the 10-Year Rate



Panel C: Implied Quantiles for the 20-Year Rate

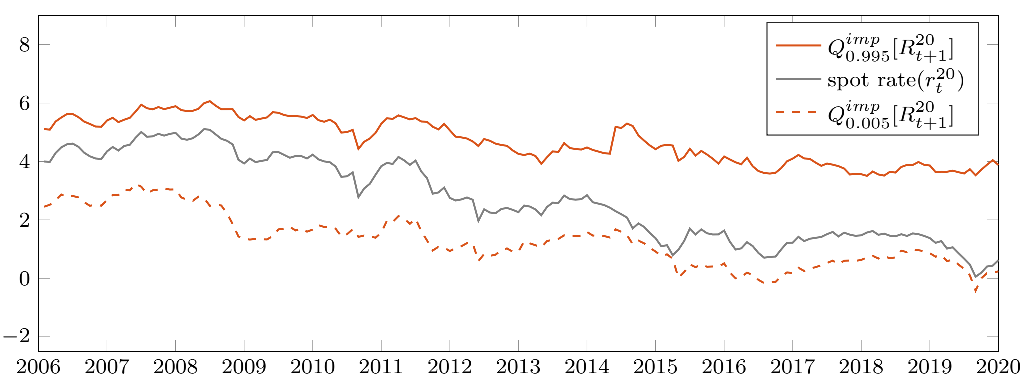


Figure 2: This figure shows the 0.5%- and 99.5%-quantiles of the one-year ahead interest rate. The quantiles are calculated based on physical option-implied distributions that are extracted with two-state mixture models and a quadratic (log-)pricing kernel. We show quantiles for the 5-, 10- and 20-year EUR rates as well as the corresponding spot rates. Numbers are in per cent.

downward and upward shocks that are defined as 0.5%- and 99.5%-quantiles of the change $\Delta R^n_{t,1}$ in the spot rate over one year, i.e., we consider $Q_{0.005}[\Delta R^n_{t,1}]$ and $Q_{0.995}[\Delta R^n_{t,1}]$.[44] Summary statistics for the upward and downward shocks to the 10-year rates from 01/2006 until 12/2019 are reported in Table 1 and the evolution of these shocks over time is shown in Figure 3. Similar statistics for the 5- and 20-year rates are provided in the Appendix. To improve comparability of the results, descriptive statistics for the downward shocks are reported in absolute values.

The average option-implied upward shock for the 10-year rate is equal to 2.01%. Over our sample period, the implied upward shocks range between 0.97% and 2.97%. For the implied downward shocks, we document a time-series average of 1.39% and a range between 0.63% and 2.83%. The time-series variation of our quantile estimates mirrors the asymmetries of the implied spot rate distributions documented in Figure 2. As can be seen from Figure 3, the magnitude of downward interest rate shocks is much larger during the first half of our sample period, whereas the upward shocks are larger during the low interest rate period starting after the financial crisis.

Next, we compare our option-implied risk estimates to shocks that are calculated according to the Solvency II standard formula (see Appendix A). The Solvency II upward shocks range between 1.00% and 2.12% with a time-series average of 1.26%. Accordingly, the average and the range of the Solvency II upward shocks are smaller than the corresponding values for our option-implied estimates. Differences are particularly pronounced during the second half of our sample period, when the option implied estimates are often more than twice as large as the Solvency II upward shocks. For the downward shocks, we find that the magnitude of the regulatory shocks is always smaller than the magnitude of the implied quantile estimates with the average of the Solvency II values being only 0.70%. In particular, the magnitude of the Solvency II downward shocks has been extremely small since 2015 and has reached its lower bound of *zero* in the second half of 2019, when the 10-year spot rate became negative. Finally, the monthly changes in the upward shocks for our implied method and the two Solvency II benchmarks are negatively correlated whereas we document positive correlations for the downward shocks.

As a second benchmark for our option-implied risk estimates, we consider the revised Solvency II methodology (EIOPA, 2020). First, we observe that the revised shocks are almost always larger

[44] We assume a time-invariant spread between the spot rate and the corresponding swap rate in (20). This implies that the *change* in the one-year spot rate $\Delta R^n_{t,1}$ is equal to the *change* in the swap rate $\Delta S^n_{t,1}$.

Table 1: Interest Rate Risk – 10-Year Rate

	avg	std	min	q25	med	q75	max
Panel A: Upward Shocks							
imp	2.01	0.57	0.97	1.67	2.10	2.47	2.97
Solvency II	1.26	0.35	1.00	1.00	1.00	1.52	2.12
Solvency II*	1.73	0.45	0.97	1.32	1.64	2.13	2.56
norm-month	1.70	0.13	1.44	1.63	1.74	1.80	1.85
EGARCH	1.42	0.18	0.99	1.31	1.39	1.51	2.19
Vasicek	0.92	0.26	0.35	0.91	1.02	1.07	1.27
Panel B: Downward Shocks (abs.)							
	avg	std	min	q25	med	q75	max
imp	1.39	0.62	0.63	0.85	1.16	1.83	2.83
Solvency II	0.70	0.47	0.00	0.28	0.61	1.12	1.56
Solvency II*	1.52	0.60	0.50	0.97	1.40	2.05	2.63
norm-month	1.70	0.13	1.44	1.63	1.74	1.80	1.85
EGARCH	2.13	0.24	1.67	1.96	2.12	2.23	2.96
Vasicek	0.53	0.10	0.12	0.49	0.52	0.56	0.91
Panel C: Upward Shocks - Correlations (chg.)							
	(a)	(b)	(c)	(d)	(e)	(f)	
(a) imp	1.00						
(b) Solvency II	-0.46	1.00					
(c) Solvency II*	-0.58	0.74	1.00				
(d) norm-month	0.33	-0.29	-0.19	1.00			
(e) EGARCH	0.22	-0.15	-0.04	0.20	1.00		
(f) Vasicek	0.14	-0.19	-0.11	0.39	0.17	1.00	
Panel D: Downward Shocks - Correlations (chg.)							
	(a)	(b)	(c)	(d)	(e)		
(a) imp	1.00						
(b) Solvency II	0.35	1.00					
(c) Solvency II*	0.33	0.99	1.00				
(d) norm-month	-0.06	0.19	0.19	1.00			
(e) EGARCH	0.15	-0.29	-0.31	-0.21	1.00		
(f) Vasicek	-0.08	0.09	0.09	0.38	-0.10	1.00	

This table reports summary statistics for upward and downward shocks to the 10-year spot rate calculated at the end of each month of our sample period between 2006 and 2019. The shocks are constructed to be in line with the 99.5%-VaR objective, i.e., they correspond to the 0.5%- and the 99.5%-quantiles of the change in the spot rate over one year. We report results for our option-implied approach relying on two-state mixtures and a quadratic (log-)SDF (imp), the current Solvency II standard formula (Solvency II) and the revised Solvency II methodology (Solvency II*). Furthermore, we include a Gaussian benchmark based on monthly data (norm-month). Additionally, we report results for simulation-based EGARCH quantile forecasts with normally distributed innovations as well as tail risk forecasts based on the Vasicek model. We refer to Section IV in the Online Appendix for details on the benchmark methods. For each methodology, we report the time-series average (avg), the standard deviation (std), the minimum (min), the 25%-quantile (q25), the median (med), the 75%-quantile (q75) and the maximum (max). We also report correlations for monthly *changes* in upward and downward shocks. Statistics for the downward shocks are reported in absolute values. Shocks are measured in percentage points.

Figure 3: Panel A shows the upward and downward shocks for the 10-year spot rate according to the option-implied approach (implied), the current Solvency II standard formula (Solvency II) and the recent revision of the standard Solvency II methodology (Solvency II*). Formally, these shocks correspond to the 0.5%-quantiles and the 99.5%-quantiles of the change in the interest rate over a one-year horizon. The swaption-implied estimates are based on two-state mixture models and a quadratic (log-)SDF. Panel B compares the option-implied risk estimates with a Gaussian benchmark based on monthly data (norm-month) and simulation-based quantile estimates based on an EGARCH model (see Appendix IV for the details). Numbers are in percentage points.

than the shocks according to the current Solvency II rules. On average, the differences to our option-implied methodology therefore become smaller. The impact of the revised proposal is particularly large for interest rate decreases: the average downward shock more than doubles from 0.70% to 1.52%. And the new downward shocks are overall much closer to our option-implied quantiles as can be seen from Figure 3.

Motivated by the SST methodology (Finma, 2017), we include quantile estimates that are derived from 10 years of monthly interest rate changes with a normality assumption and square root of time scaling.[45] The average level of the resulting symmetric quantile estimates is 1.70%, which is also closer to the option-implied counterparts than to the results obtained from the current Solvency II standard formula. However, the symmetry and the low responsiveness to changing market conditions cause substantial differences to our option-implied estimates over time.

We also consider benchmark forecasts derived from an EGARCH-specification.[46] Differences to our implied benchmark have become particularly apparent since 2011. During the second half of our sample period, EGARCH upward shocks are lower than our option-implied estimates, whereas the EGARCH downward shocks are much stronger than the corresponding option-implied risk forecasts.[47]

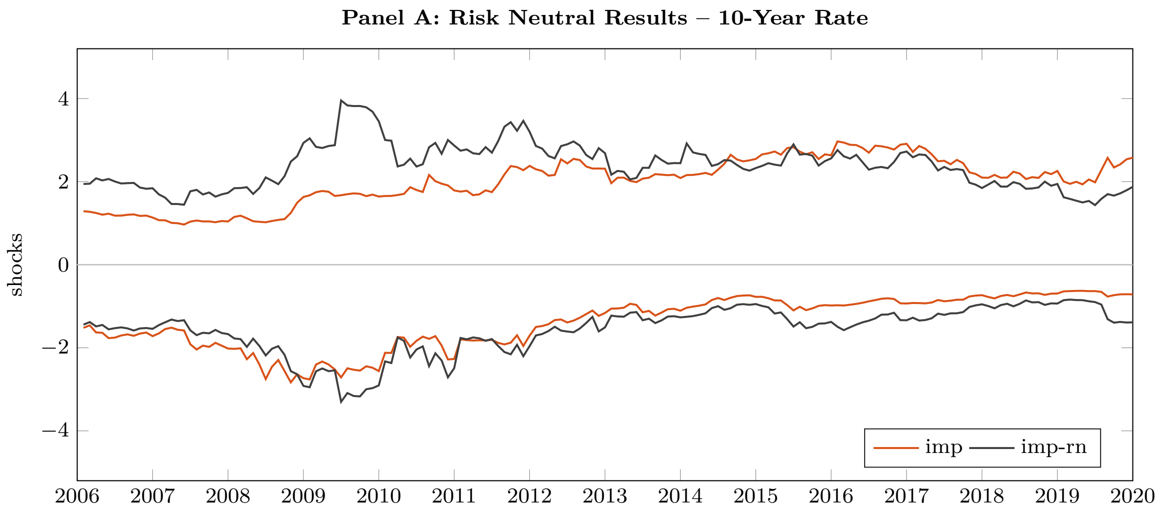
Finally, we derive tail risk forecasts based on the Vasicek (1977) model. For this model, we find the lowest average upward (0.92%) and downward shocks (0.53%). Vasicek-based upward shocks are close to the Solvency II shocks for the second half of our sample period.

4.3 Risk-Neutral Results and Robustness

In this section, we investigate the impact of the measure change on our results. Panel A of Figure 4 compares option-implied quantile estimates for the change in the 10-year rate under the forward risk-neutral measure and under the physical measure. It reveals that the effect of the measure change on the left and the right tail can be rather different and that the overall impact of the measure change is time-varying. In particular, we document a strong effect on the probabilities of rate increases during the first half of our sample period.

Given this importance of the measure change for the probability mass in the tails of the phys-

[45] Details on the implementation of our benchmark estimators are presented in Appendix IV.
[46] Details are provided in Section IV of the Online Appendix.
[47] Besides the EGARCH specification, we also analyzed other GARCH-type models and found similar results.



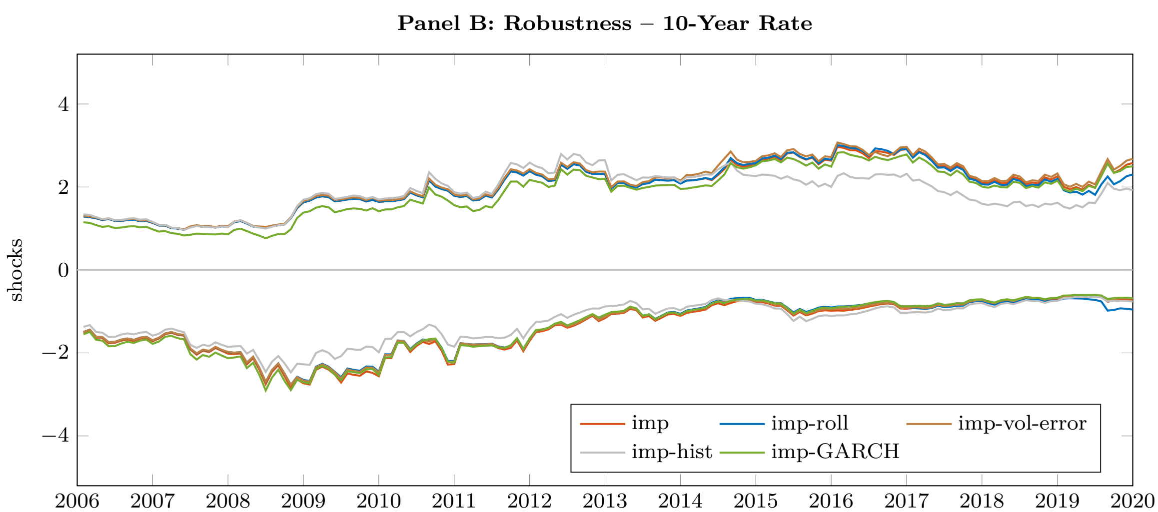


Figure 4: This figure shows the option-implied upward and downward shocks to the 10-year spot rate calculated at the end of each month of our sample period between 2006 and 2019. The shocks are constructed to be in line with the 99.5%-VaR objective, i.e., they correspond to the 0.5%- and the 99.5%-quantiles of the change in the spot rate over one year. Panel A compares the option-implied physical shocks (imp) with the corresponding shocks under the risk-neutral measure (imp-rn). Panel B shows the results of our robustness checks where we apply the following variations with respect to the determination of the SDF parameters based on (16). First, we replace the growing estimation window by a rolling window (imp-roll). Second, we implement the moment-matching technique based on volatilities instead of variances (imp-vol-error). Third, we replace the baseline EGARCH variance forecasts by simple model-free variance forecasts based on daily data (imp-hist) and by simulation-based forecasts derived from GARCH(1,1)-models. Numbers are in percentage points.

ical distribution, we finally examine the robustness of our methodology for calibrating the SDF parameters γ_t and δ_t. First, we use a rolling estimation window with 84 observations instead of the growing window procedure. Second, we apply the moment matching in equation (16) on the level of volatilities instead of variances. Third, we replace the EGARCH benchmark variance forecasts by model-free realized variance forecasts based on daily data and by variance forecasts from GARCH(1,1)-models.[48] As can be seen from Panel B of Figure 4, our risk forecasts for the 10-year rate are relatively stable under these variations. The average upward (downward) shocks for the 10-year rate only range between 1.85 and 2.04 (1.27 and 1.40) across different implementations of the moment matching.[49]

5 Results for Equity Risk

5.1 Data and Calibration

We use options on the Eurostoxx 50, the S&P 500 and the FTSE 100. We collect prices for all available strikes at observation dates between 01/2006 until 12/2019 with a monthly frequency from Datastream. For each index and each date, we select two option cross sections with maturities $\tau_1 < 1$ and $\tau_2 > 1$.[50] We use Bloomberg dividend yield estimates and EUR-, USD- and GBP-Libor rates from Datastream.[51,52]

We then apply the following standard filters across the strike range: (i) We only keep out-of-the-money (OTM) options. (ii) We delete options violating no-arbitrage constraints. (iii) We delete contracts with extreme moneyness levels. Following Andersen et al. (2015), we measure the moneyness of a contract in terms of its at-the-money Black-Scholes implied volatility, i.e. we define

$$m = \frac{\log(Y/F_{t,\tau})}{\tau\, iv_{\text{bs}}^{atm}}, \tag{23}$$

where Y and τ are the strike price and the time to maturity of the given option contract, $F_{t,\tau}$ is

[48] We again refer to Section IV in the Online Appendix for more details on the benchmark variance forecasts.

[49] Summary statistics for the corresponding tail risk time series are presented in Table V.2 in the Online Appendix.

[50] For the FTSE, we remove the June cross sections if the time to maturity is larger than 1 year since the number of prices and the moneyness range are often relatively small.

[51] The Libor rates only exist for maturities ≤ 1 year. For options with maturities longer than 1 year, we use the 1-year rate.

[52] Other studies using long-term equity options include Collin-Dufresne et al. (2012) and Bakshi et al. (2000, Table 1).

the time-t price of a futures contract with maturity $t + \tau$ and iv_{atm} is the Black-Scholes implied volatility of the at-the-money option contract. Using this moneyness measure, we restrict our sample to options with $-8 \leq m \leq 5$. (iv) Similar to Andersen et al. (2017), we also delete options that violate no-arbitrage relations across the strike range.

After applying these filters, we have on average 65 prices per calibration date and cross-section for the Eurostoxx 50, 60.5 prices for the S&P 500 and 48.5 prices for the FTSE. In total, we use more than 50 000 option prices for the analyses. Across all months in our sample period, the average minimum moneyness as defined in equation (23) is $m = -5.2$ for the FTSE 100 and roughly -7.0 for the Eurostoxx 50 and the S&P 500. Figure V.3 in the Online Appendix illustrates some characteristics of our option data set for the Eurostoxx 50.

Finally, we use daily Eurostoxx 50, S&P 500 and FTSE 100 log-returns from Datastream in order to estimate the SDF parameter γ_t based on the methodology explained in Section 3.3. As for the interest rate case, we apply a growing estimation window and use in-sample estimates for the first half of our sample. The magnitudes of the γ_t estimates are relatively similar for the three indices. The Eurostoxx 50 and the S&P 500 estimates both range between 1.6 and 2.0. The FTSE 100 estimates are slightly higher and range between 2.1 and 2.5.

5.2 Option-Implied Estimates and Benchmarks

In our equity risk analysis, we focus on the left tail and report equity shocks that are consistent with the 99.5%-VaR, i.e., the 0.5%-quantile of the (discrete) return distribution multiplied by minus one. Our option-implied quantile estimates and selected benchmarks are presented in Figure 5. Summary statistics are reported in Table 2.

We start with a comparison of the option-implied tail risk estimates for the three markets depicted in Panel A of Figure 5. We find the highest average estimate for the Eurostoxx 50 index (47.76%) followed by the S&P 500 (45.52%). The average implied VaR for the FTSE 100 is somewhat lower with an average value of 42.97%.[53] The option-implied VaR forecasts exhibit a substantial variation over time with values ranging from 38.41% to 62.33%. Not surprisingly, changes in the implied tail risk levels are positively correlated across markets with all indices attaining their lowest risk levels in 2006 and their highest levels during the financial crisis.

[53] The FTSE 100 has a relatively large exposure to conservative sectors such as consumer staples and health care.

Table 2: Equity Risk

Panel A: Solvency II							
	avg	std	min	q25	med	q75	max
Solvency II without SA	39.00	0.00	39.00	39.00	39.00	39.00	39.00
Solvency II with SA	38.41	6.04	29.00	33.88	38.66	42.96	49.00
Panel B: Eurostoxx 50							
	avg	std	min	q25	med	q75	max
imp	47.76	5.14	38.41	43.99	47.10	51.01	62.33
imp-SA	47.17	3.65	37.44	44.47	47.36	49.97	55.32
norm-month	38.71	2.37	32.94	36.84	38.25	41.02	41.67
EGARCH	46.03	3.13	38.23	44.74	46.28	48.09	53.64
GARCH-emp	52.43	5.71	42.81	49.48	51.61	53.75	74.85
Panel C: S&P 500							
	avg	std	min	q25	med	q75	max
imp	45.52	5.79	31.01	42.51	45.17	48.83	62.47
imp-SA	44.92	3.55	37.66	42.27	44.37	47.47	54.28
norm-month	32.96	1.68	28.42	32.19	33.09	34.05	35.47
EGARCH	42.94	4.90	36.12	39.59	41.85	45.16	63.33
GARCH-emp	47.79	12.52	34.81	39.15	43.43	51.72	96.33
Panel D: FTSE 100							
	avg	std	min	q25	med	q75	max
imp	42.97	5.01	32.55	39.36	42.13	45.62	58.42
imp-SA	42.38	3.46	32.58	40.23	41.85	44.68	51.08
norm-month	31.76	1.67	26.70	30.83	32.11	32.65	34.36
EGARCH	39.18	2.86	32.50	37.75	38.96	40.39	49.66
GARCH-emp	42.97	6.17	35.20	39.63	41.42	43.93	77.74
Panel E: Correlations (chg.)							
	(a)	(b)	(c)	(d)	(e)	(f)	(g)
(a) Solvency II	1.00						
(b) imp-Eurostoxx	-0.63	1.00					
(c) imp-S&P	-0.60	0.81	1.00				
(d) imp-FTSE	-0.58	0.72	0.72	1.00			
(e) imp-SA-Eurostoxx	0.27	0.56	0.37	0.26	1.00		
(f) imp-SA-S&P	0.29	0.34	0.59	0.28	0.71	1.00	
(g) imp-SA-FTSE	0.10	0.37	0.40	0.75	0.53	0.58	1.00

This table reports summary statistics for the monthly time series of equity shocks from 2006 until 2019. The shocks are constructed to be in line with the 99.5%-VaR objective over a one-year horizon. We compare the option-implied estimates based on three-state mixture models and a linear (log-)SDF (imp) with the Solvency II standard formula. We also analyze option-implied shocks combined with the Solvency II symmetric adjustment (imp-SA). Additionally, the following time series benchmarks are considered (without SA): a Gaussian benchmark with monthly data (norm-month) as well as forecasts from GARCH- and EGARCH-models estimated with daily data. For the latter, the simulation is based on the empirical distribution of the fitted innovations (GARCH-emp). For each series, we report the average (avg), the standard deviation (std), the minimum (min), the 25%-quantile (q25), the median (med), the 75%-quantile (q75) and the maximum (max). These numbers are in per cent. Panel E shows correlations for monthly *changes* in the tail risk estimates.

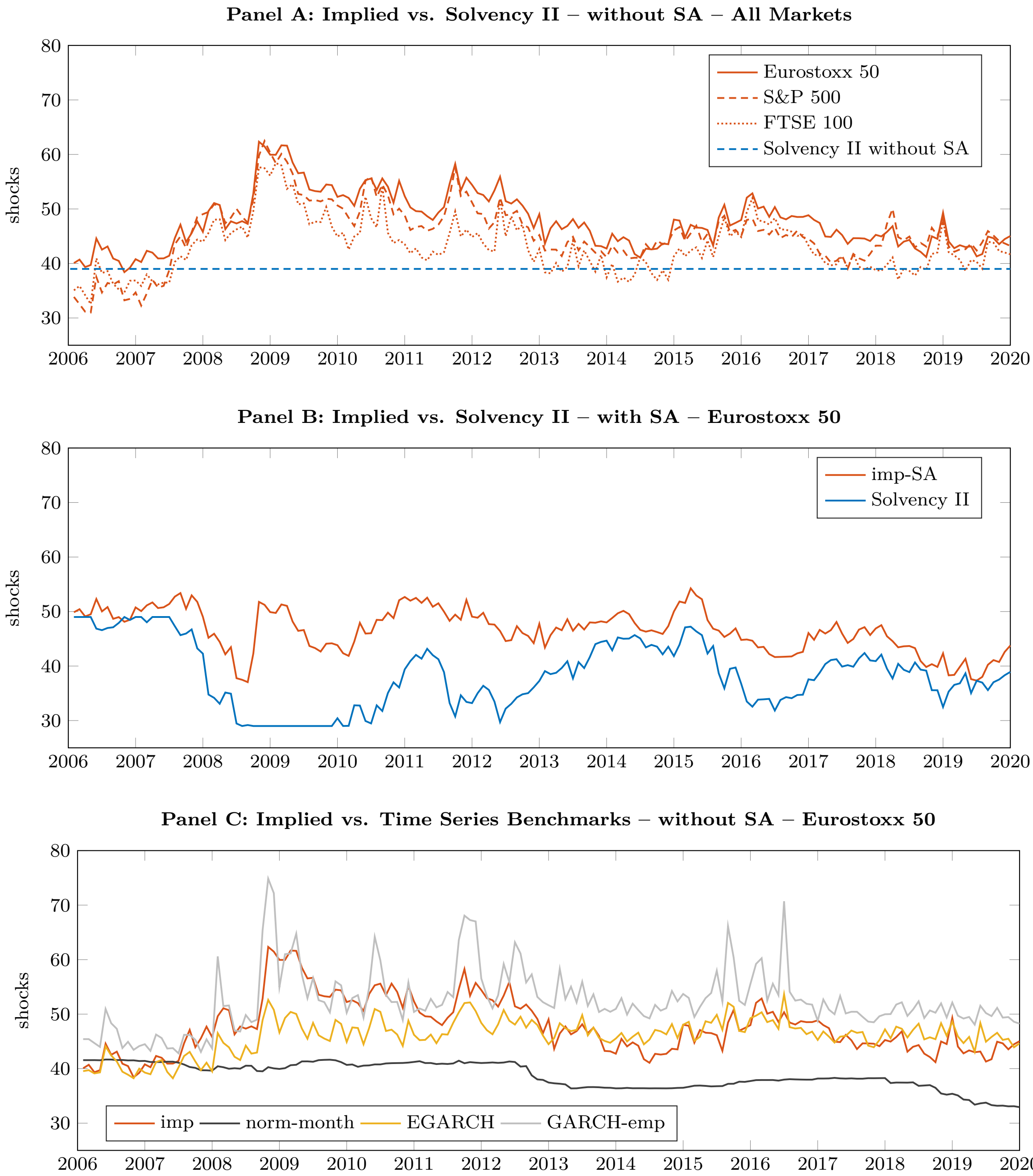


Figure 5: The figure depicts equity return shocks that are consistent with the 99.5%-VaR over one year. Panel A shows the Solvency II baseline shock without the symmetric adjustment and the shocks according to our option-implied approach for the Eurostoxx 50, the S&P 500 and the FTSE 100. Panel B shows the option-implied estimates for the Eurostoxx 50 combined with the symmetric Solvency II adjustment as well as the shocks according to the Solvency II standard formula (with SA). Panel C compares the option-implied Eurostoxx 50 estimates with the following time-series methods: A Gaussian benchmark based on monthly data (norm-month) and simulation-based (E)GARCH-benchmarks. Numbers are in percentage points.

Overall, the option-implied shocks are mostly larger than the static Solvency II baseline shock of 39.00% as can be seen from Panel A in Figure 5. The average differences lie between 3.97 percentage points for the FTSE and 8.76 percentage points for the Eurostoxx 50. However, the differences attain almost 25 percentage points during the financial crisis.

The relatively large time-variation of the implied equity shocks is substantially reduced by applying the Solvency II symmetric adjustment factor. For the implied shocks, this counter-cyclical adjustment dampens the increase in tail risk during crisis periods and it raises the shock levels during calm periods.[54] Interestingly, the *adjusted* option-implied estimates are much less volatile than the (adjusted) Solvency II shocks according to the standard formula as can be seen from the standard deviations reported in Table 2. By construction, the adjustment has almost no impact on the time-series averages.

Inspired by the SST, we again include the Gaussian benchmark based on rolling estimation windows with monthly data.[55] These benchmark estimates are on average around ten percentage points lower than the option-implied estimates. Furthermore, by construction, the Gaussian estimates are more stable over time.

Comparisons of the option-implied tail risk measures with GARCH-based risk forecasts depend on the specific implementation of the time-series models. Whereas the average tail risk forecasts from EGARCH models with Gaussian innovations are somewhat lower than the average option-implied estimates, the average VaR-levels predicted with Filtered Historical Simulation (Barone-Adesi et al., 1998) are greater than or equal to the option-implied estimates. Furthermore, the GARCH-based long-term tail risk estimates also exhibit a substantial degree of variation over time, which can be rather extreme for the case of Filtered Historical Simulation.

5.3 Robustness and Risk-Neutral Results

Finally, we present results for selected alternative option-implied equity shocks in Figure 6 for the Eurostoxx 50.[56] A comparison of our results with *risk-neutral* three-state mixture models shows that the measure change is an important component of our methodology for long-term equity tail

[54] For the Eurostoxx 50, we no longer observe the maximum shock during the financial crisis. Instead, the maximum level of 55.32% is achieved in 2015.

[55] See again Section IV in the Online Appendix for details on the construction of the benchmark forecasts.

[56] The corresponding summary statistics are presented in Table V.3 in the Online Appendix.

risk.

To check the robustness of our results with respect to the specific models that we use to describe the risk-neutral distributions, we implement our methodology based on two-, four- and five-state mixture models (instead of the three-state baseline specification). Similar to the interest rate case, we also analyze the robustness of our SDF calibration by varying its implementation as follows. First, we implement the matching algorithm based on a rolling instead of a growing estimation window to identify the SDF parameter. Second, we replace variances by volatilities in equation (16). Third, we use absolute squared variance errors instead of relative squared errors. Fourth, we replace the EGARCH benchmark variance forecasts by model-free realized variance forecasts from daily data and by GARCH(1,1) variance forecasts. We also present the results for the more general quadratic specification of the (log-)SDF and, finally, we relax our option moneyness filter by decreasing the lower bound from -8 to -15. Our tail risk estimates are stable with respect to these variations. We only find somewhat higher average shocks for the more restrictive two-state specification (52.23%). For the remaining specifications, the average 99.5%-VaR estimates range between 46.57% and 49.02% and the time series are highly correlated (see Figure 6).

6 Solvency Capital Requirements

6.1 A Forward-Looking Internal Model

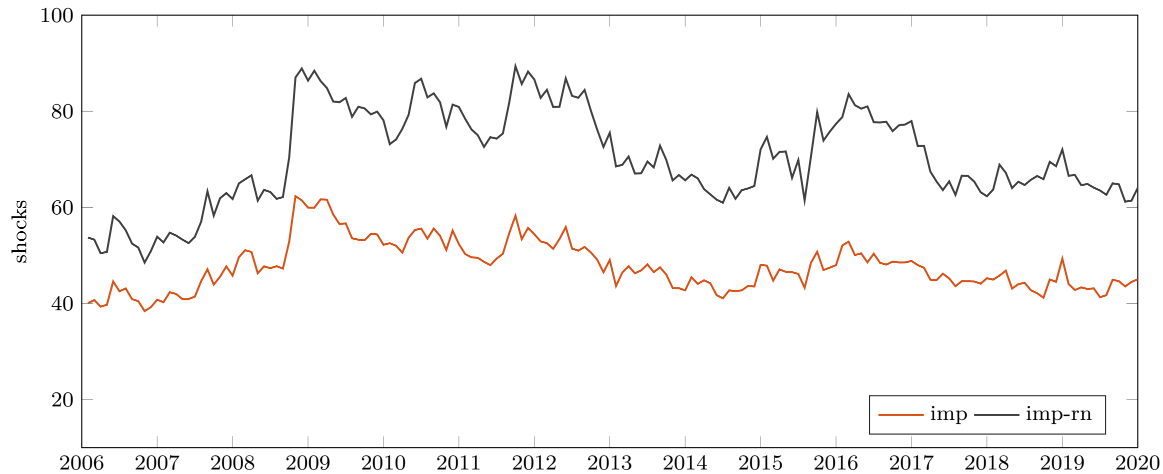
To understand the economic significance of the differences that we documented in the previous sections, we now integrate our option-implied shocks in an (partial) internal model and compare the resulting aggregated SCRs to our benchmark techniques.

Therefore, we propose to replace the Solvency II equity and interest rate shocks presented in equation (5) by our option-implied counterparts. Formally, the SCR for module i is then determined based on

$$SCR_i = -\Delta BOF \,|\, \textit{implied-shock}_i, \tag{24}$$

where $\textit{implied-shock}_i$ indicates the shock scenario according to our option-implied tail risk estimates. For equity risk, the option-implied 99.5%-VaR estimates according to (22) replace the

Panel A: Risk Neutral Results – Eurostoxx 50



Panel B: Robustness – Eurostoxx 50

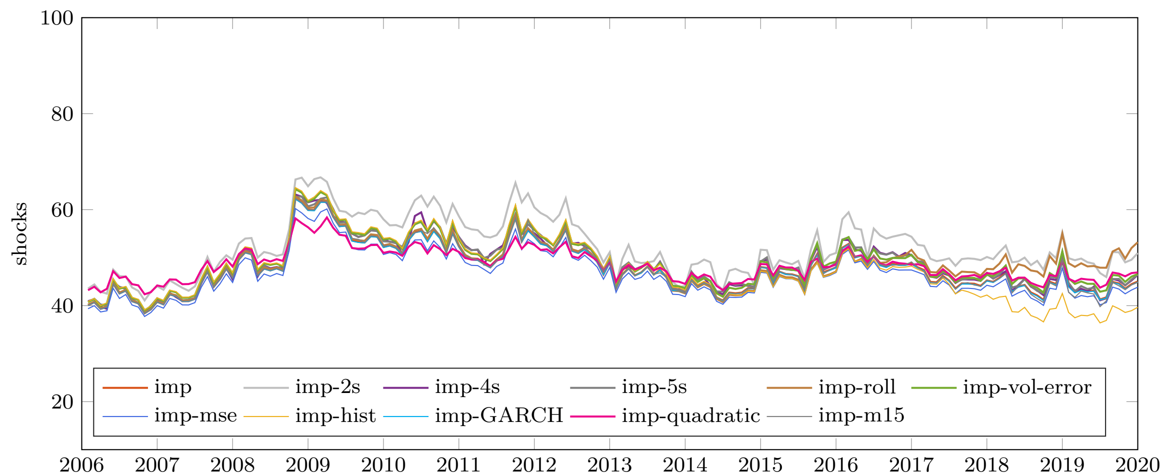


Figure 6: This figure shows the time-series of Eurostoxx 50 shocks from 2006 until 2019. The shocks are constructed to be in line with the 99.5%-VaR objective over a one-year horizon. We report the option-implied baseline results for the three-state mixture model and a linear (log-)SDF. Panel A compares risk-neutral and physical results. Panel B presents the results of our robustness analyses. We show the results for two-, four- and five-state mixture models. Furthermore, we consider the following variations with respect to the determination of the SDF parameter based on equation (16). First, we we estimate the SDF-parameter based on a rolling estimation window instead of a growing window (imp-roll). Second, we implement the moment-matching technique based on volatilities instead of variances (imp-vol-error). Third, we replace relative variance errors by absolute variance errors (imp-mse). Fourth, we replace the EGARCH-based variance forecasts by model-free realized variance forecasts (imp-hist) and by GARCH(1,1) variance estimates (imp-GARCH). Furthermore, we present the results for the more general quadratic specification of the (log-)SDF (imp-quadratic) and we present the results for the moneyness filter $-15 < m < 5$ (imp-m15). Numbers are in percentage points.

Solvency II shocks. For interest rate risk, the swaption-implied quantiles are used.[57]

For all other sub-modules, the standard formula can be applied to determine the respective capital requirements. Finally, the overall market risk SCR can be calculated based on the square root aggregation formula (4). This procedure is in line with the default integration technique proposed in the Delegated Regulation (European Commission, 2014, Article 239).

As an additional benchmark, we also include a partial internal model with shocks under the assumption of normally distributed equity returns and interest rate changes as proposed in the context of the SST.

6.2 Market Risk Solvency Capital Requirements

In this section, we compare overall capital requirements for market risk for two stylized life insurance balance sheets:

Balance Sheet I				Balance Sheet II			
Stocks	5	Basic own funds	12	Stocks	10	Basic own funds	12
Bonds 5Y	15	Liabilities (10Y)	88	Bonds 5Y	30	Liabilities (10Y)	88
Bonds 10Y	80			Bonds 10Y	60		

For ease of interpretation, we set the total market value of the balance sheets to 100. In line with Braun et al. (2017), we assume a value of 12% for the ratio of free to total assets, which is defined as the value of basic own funds divided by the balance sheet total. The bond positions are approximated by positions in 5- and 10-year (default-free) zero-coupon bonds and the liabilities are approximated by a 10-year zero-coupon bond. These approximations can be justified by a duration mapping technique (Jorion, 2007) and our choices are largely consistent with the asset and liability durations considered by Braun et al. (2017). Furthermore, we assume that the stock position corresponds to an investment in the Eurostoxx 50.

The "low-risk" balance sheet I is characterized by a stock ratio of 5%, which is in line with current estimates for the German life insurance market.[58] Furthermore, the amount invested in the medium-term 5-year zero-coupon bond is set to 15. For balance sheet II, we assume a stock ratio

[57] Note that interpolation techniques could be used if shocks are required for maturities at which swaption quotes are not available.

[58] See https://www.gdv.de/de/zahlen-und-fakten/versicherungsbereiche/kapitalanlagen-24042.

twice as high and we also double the amount invested in the medium-term bond. Doing so, we increase the mismatch between the duration of assets and liabilities, that is, we widen the duration gap.

The capital requirement for equity risk at time t is given by

$$SCR_{eq} = v_{stocks} \cdot shock_{eq,t}, \tag{25}$$

where v_{stocks} is the amount invested in stocks and $shock_{eq,t}$ is the time-t equity shock.

The capital requirement for interest rate risk is given by $\max\{SCR_{int}(up); SCR_{int}(down)\}$ with

$$SCR_{int}(up/down) = v_5 + v_{10} - \left(x_{5,t}\,(1 + r_t^{5,up/down})^{-5} + x_{10,t}\,(1 + r_t^{10,up/down})^{-10}\right), \tag{26}$$

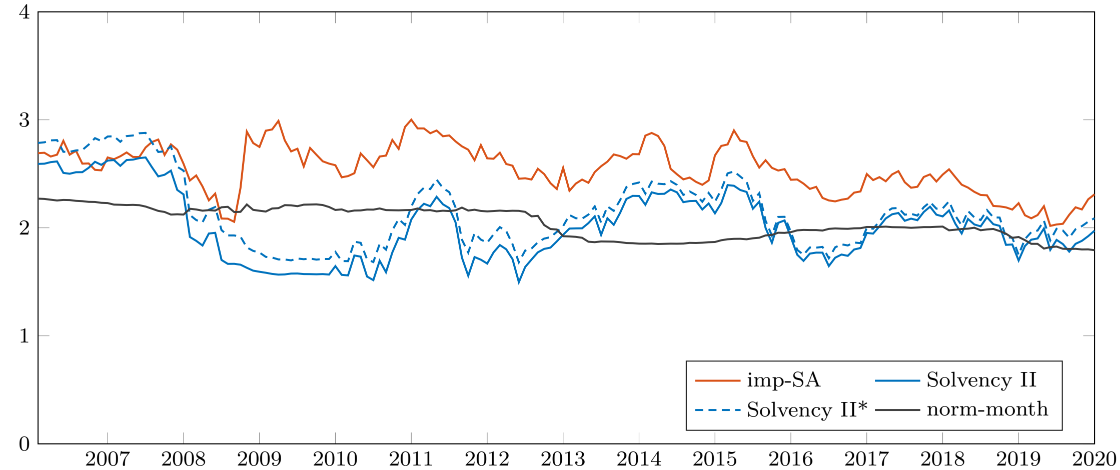
where $r_t^{n,up/down}$ is the shocked n-year interest rate at time t, v_n is the (net) amount invested in the n-year bond and $x_{n,t} = v_n \cdot (1 + r_t^n)^n$ is the (net) number of n-year zero-coupon bonds with payoff EUR 1 at time t. The overall market risk capital requirement SCR_{market} is then calculated based on the square root aggregation rule (4).

For our analysis, we assume the balance sheets to be equal at each forecasting date. Descriptive statistics for the monthly market risk capital requirements between 01/2006 and 12/2019 are reported in Table 3 and the corresponding time series are presented in Figure 7. Since we set the balance sheet total to 100, *all numbers are in percent of total assets.*

We first discuss the results for balance sheet I. The option-implied capital requirements for market risk range between 2.02 and 3.00 (per cent of total assets). The average implied SCR is given by 2.54. The average Gaussian estimate (2.05) and the average Solvency II requirement (2.01) are lower. The benchmarks clearly show a different variation over time. Whereas the Gaussian SCRs do not fluctuate much, we observe a pronounced time variation for the Solvency II capital requirements and for our option-implied approach. However, option-implied capital requirements and the Solvency II benchmarks often move in opposite directions.

The market risk capital requirements for the second balance sheet are higher reflecting the larger exposure to stock market and interest rate risk. Implied market risk SCRs range between 4.51 and 8.72. The average implied capital requirement is equal to 6.21 and is thus more than

Panel A: Market Risk Capital Requirements – Balance Sheet I



Panel B: Market Risk Capital Requirements – Balance Sheet II

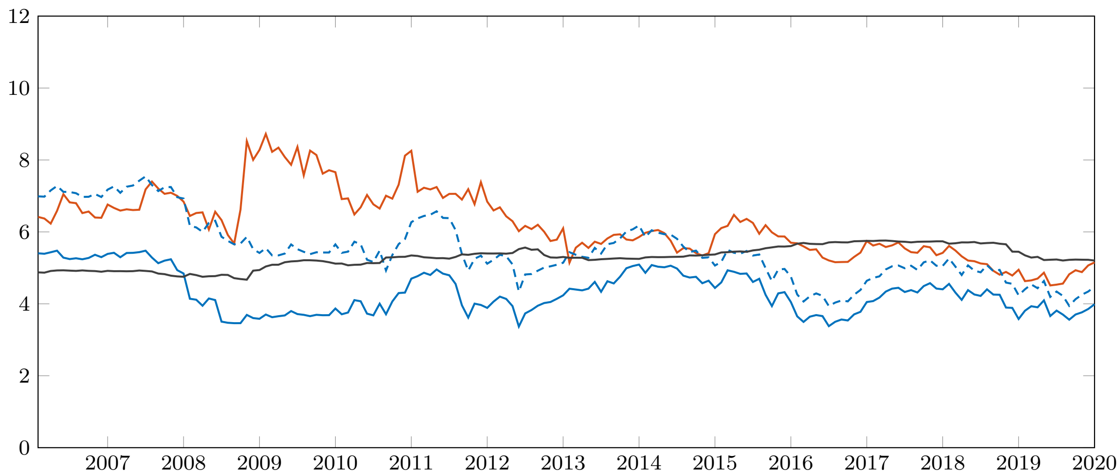


Figure 7: The figure shows the time-series of market risk solvency capital requirements for balance sheet I (Panel A) and balance sheet II (Panel B). The balance sheet total is always set to 100 and the ratio of free to total assets equals 12%. Results are reported for our option-implied approach taking the standard formula's symmetric equity adjustment into account (imp-SA), the normal distribution benchmark based on monthly data (norm-month) and Solvency II. For Solvency II, we distinguish between the current standard formula and the revised method based on the newly proposed interest rate shocks (Solvency II*).

Table 3: Market Risk Capital Requirements

	avg	std	min	q25	med	q75	max
Panel A: Balance Sheet I							
	avg	std	min	q25	med	q75	max
imp-SA	2.54	0.22	2.02	2.40	2.55	2.70	3.00
Solvency II	2.01	0.32	1.50	1.75	1.99	2.23	2.65
Solvency II*	2.14	0.33	1.68	1.88	2.09	2.33	2.88
norm-month	2.05	0.14	1.79	1.91	2.01	2.17	2.27
Panel B: Balance Sheet II							
	avg	std	min	q25	med	q75	max
imp-SA	6.21	0.95	4.51	5.54	6.07	6.83	8.72
Solvency II	4.32	0.59	3.37	3.79	4.25	4.79	5.48
Solvency II*	5.49	0.89	3.94	4.94	5.38	5.96	7.54
norm-month	5.29	0.30	4.67	5.10	5.29	5.51	5.76

This table reports descriptive statistics for the monthly time-series of market risk solvency capital requirements for balance sheet I (Panel A) and balance sheet II (Panel B). The balance sheet total is equal to 100 and the ratio of free to total assets equals 12%. Results are reported for our option-implied approach taking the standard formula's symmetric equity adjustment into account (imp-SA), the normal distribution benchmark based on monthly data (norm-month) and Solvency II. For Solvency II, we distinguish between the current standard formula and the revised method based on the newly proposed interest rate shocks (Solvency II*; see EIOPA, 2020, p. 31 f.). For each series, we report the average (avg), standard deviation (std), minimum (min), 25%-quantile (q25), median (med), 75%-quantile (q75) and the maximum (max).

40% higher than the average Solvency II capital requirement (4.32). Compared to balance sheet I, we document larger differences between the current Solvency II approach and the proposal for the revised standard formula. As for the first balance sheet, the implied SCRs are highest during the financial crisis and remain above average in the years after. In contrast, Solvency II SCRs are below their average in this period.[59] The average Gaussian SCR is equal to 5.29 and thus between the average requirements according to the option-implied approach and the current Solvency II standard formula.

7 Conclusion

This paper examines the measurement of long-term market tail risk. For this purpose, we propose a new option-based methodology and implement this approach in the Solvency II framework for interest rate and equity risk. We compare our option-implied VaR forecasts to shocks derived from

[59]Note that we do *not* account for equity losses in this period because we consider equal balance sheets at each observation date.

the Solvency II standard formula and to estimates from traditional statistical methods relying on past risk factor changes.

Our general methodology combines a normal mixture model for the risk-neutral distribution of risk factor changes with a quadratic approximation of the (log-)projected SDF. We extract the required distribution parameters from current option prices and rely on a variance matching technique for calibrating the SDF parameters. In particular, we recover the distribution of interest rate changes from one-year into n-year swaptions and use standard index options with a time-to-maturity close to one year for the estimation of one-year equity VaRs for three important markets.

During our sample period from 2006 - 2019, the interest rate shocks according to the current Solvency II standard formula are often substantially lower than the corresponding option-implied shocks. Interestingly, our option-implied estimates are closer to the recent EIOPA proposal on revising the standard formula (EIOPA, 2020) and to shocks that are derived from Gaussian benchmarks calibrated to past interest rate changes. Similarly, the average level of Solvency II equity shocks tends to be somewhat lower than the average level of the option-implied equity VaRs. Furthermore, we document a pronounced variation in the option-implied equity tail risk estimates over time, which is correlated with the tail risk changes predicted by GARCH-type models but less extreme compared to popular implementations such as Filtered Historical Simulation.

By integrating our option-implied estimates into a partial internal model, we illustrate that the documented differences between regulatory and option-implied tail risk estimates can be economically important as they have a substantial impact on solvency ratios in our examples.

Our findings suggest that option-implied methods can help to address a key challenge in determining capital requirements for market risk over long holding periods, which is the lack of sufficient data of price changes over longer horizons. Given this advantage, option-implied long-term risk estimates might add valuable insights for recalibrating the market capital requirements of the Solvency II standard formula. Furthermore, the proposed technique might be an interesting alternative for internal risk management practices as well as for the Own Risk and Solvency Assessments (ORSA). And finally, time-varying option-implied risk estimates can potentially improve *forward-looking* investment strategies of insurance companies and pension funds.

A Solvency II: Interest Rate and Equity Risk

In this section, we provide more details on the equity risk and the interest rate risk sub-module of the Solvency II standard formula.

The rules for the equity risk module are set out in the Articles 168-173 of the Delegated Regulation (European Commission, 2014).[60] There is a distinction between type 1 and type 2 equities. All equities that are listed in countries of the European Economic Area or the OECD are assigned to type 1. In this paper, we only consider type 1 equity investments that are neither of strategic nature nor classified as long-term investments.[61] For such exposures, the capital requirement corresponds to the loss in basic own funds that results from an instantaneous equity price shock, whose magnitude is $39\% + SA_t$, where SA_t is the time dependent symmetric adjustment factor.

The calculation of SA_t is addressed in Article 172 of the Delegated Regulation. The adjustment factor is restricted to be in between -10% and $+10\%$, which implies that the downward-shocks range between 29% and 49%. EIOPA publishes current and historical symmetric adjustment factors for a representative equity portfolio.[62] The adjustment factor is constructed in a way such that the capital requirement tends to be larger in rising equity markets and lower in falling markets. It is a macroprudential tool that can help to avoid fire sales and thus a further destabilization during stress periods (EIOPA, 2014, 2017, 2018c).

As already mentioned before, the capital requirement for interest rate risk is given by

$$SCR_{int} = \max\{SCR_{int}(up), SCR_{int}(down)\}. \tag{27}$$

$SCR_{int}(up)$ is the loss in basic own funds from the upward shock in the term structure of interest rates as defined in Article 166. The upward shocks are given by

$$\Delta r_n^{up} = \max\{r_n \cdot s_n^{up}; 1.00\%\}, \tag{28}$$

[60] Note that Article 169 has been modified and Article 171a has been added recently (European Commission, 2019).

[61] For strategic and long-term investments, separate rules apply resulting in lower capital requirements. Several criteria must be fulfilled such that an investment can be classified as a strategic (European Commission, 2014, Article 171) or a long-term investment (European Commission, 2019, Article 171a).

[62] https://eiopa.europa.eu/regulation-supervision/insurance/solvency-ii-technical-information/symmetric-adjustment-of-the-equity-capital-charge

where $s_5^{up} = 55\%$, $s_{10}^{up} = 42\%$, $s_{20}^{up} = 26\%$ and r_n is the current n-year spot rate. Thus, the minimum upward shock is always equal to one percentage point.

Similar as before, $SCR_{int}(down)$ is equal to the loss in basic own funds caused by a downward shock in the risk-free term structure as described in Article 167. For the downward shocks, it holds that

$$\Delta r_n^{down} = \max\{r_n \cdot s_n^{down}; 0\}, \tag{29}$$

where $s_5^{down} = 46\%$, $s_{10}^{down} = 31\%$ and $s_{20}^{down} = 29\%$. The downward shock is thus equal to zero for negative spot rates.

Recently, EIOPA (2020) proposed adjusted shocked rates. Based on these shocked rates, the upward and downward shocks can be computed easily. For the upward shocks, we get

$$\Delta r_n^{up\star} = (r_n \cdot (1 + s_n^{up\star}) + b_n^{up\star}) - r_n = r_n \cdot s_n^{up\star} + b_n^{up\star}, \tag{30}$$

where $s_5^{up\star} = 45\%$, $s_{10}^{up\star} = 30\%$, $s_{20}^{up\star} = 25\%$, $b_5^{up\star} = 1.58\%$, $b_{10}^{up\star} = 1.05\%$ and $b_{20}^{up\star} = 0.88\%$. For the new downward shocks, it holds that

$$\Delta r_n^{down\star} = r_n - \max\{r_n \cdot (1 - s_n^{down\star}) - b_n^{down\star}; -1.25\%\}, \tag{31}$$

where $s_5^{down\star} = s_{10}^{down\star} = 40\%$, $s_{20}^{down\star} = 50\%$, $b_5^{down\star} = 0.71\%$, $b_{10}^{down\star} = 0.61\%$ and $b_{20}^{down\star} = 0.5\%$. An important difference between the current and newly proposed downward shocks is the introduction of the additive absolute shock $b_n^{down\star}$, which appears to have a strong impact in periods with low interest rates.[63]

For the empirical analysis, we use spot rates r_n that are derived from a set of EUR swap rates by bootstrapping. These bootstrapped rates are directly provided by Bloomberg. Doing so, we slightly deviate from the Solvency II approach which is based on the Smith/Wilson model.[64] Thereby, (slightly) credit-risk adjusted swap rates serve as inputs to the model which generates the final risk-free term structure. The model implicitly transforms the (adjusted) swap rates into the

[63]EIOPA (2020) does not specify adjustments for negative rates. Consequently, $r_n < 0$ implies that $\Delta r_n^{up\star} < b_n^{up\star}$ and $\Delta r_n^{down\star} < b_n^{up\star}$.

[64]We refer to Lageras and Lindholm (2016) and Jørgensen (2018) for details on the Smith/Wilson model.

corresponding spot rates very similar to the standard bootstrapping technique. By comparing the bootstrapped spot rates from Bloomberg with the risk-free rates published by EIOPA for selected dates, we find very similar values.

Finally, it is worth mentioning that we do not take volatility adjustments to the basic risk free rates into account (European Commission, 2014, Articles 49-51). Insurance companies must apply to the supervisory authority for a permission to apply these adjustments (European Commission, 2014, Annex XXI.B).

B Approximation of the Swaption Price for Mixtures

In this section, we derive approximation formulas for the prices of payer and receiver swaptions in a mixture model. For simplicity, we directly set $\tau = 1$ in this section. To obtain closed form solutions, we propose to replace the unknown future "discount rate" in (18) and (19) by the current spot swap rate. The modified payoffs are then given by

$$(S_{t+1}^n - Y_S)^+ \sum_{i=1}^{n} \frac{1}{(1+S_t^n)^i} \tag{32}$$

$$(Y_S - S_{t+1}^n)^+ \sum_{i=1}^{n} \frac{1}{(1+S_t^n)^i}, \tag{33}$$

where S_t^n is the spot swap rate at time t. Using these payoffs, the swaption prices are approximated by

$$\pi_t^{PS} \approx B_{t,1}\, \tilde{\mathbb{E}}_t[(S_{t+1}^n - Y_S)^+] \sum_{i=1}^{n} \frac{1}{(1+S_t^n)^i}, \tag{34}$$

$$\pi_t^{RS} \approx B_{t,1}\, \tilde{\mathbb{E}}_t[(Y_S - S_{t+1}^n)^+] \sum_{i=1}^{n} \frac{1}{(1+S_t^n)^i}\,. \tag{35}$$

In line with Section 3.2, we assume that the distribution of $\Delta S_{t,1}^n = S_{t+1}^n - S_t^n$ is a mixture of normals under the forward risk-neutral measure with parameters $\tilde{\boldsymbol{\pi}}_t^{\Delta s} = (\tilde{\pi}_{t,1}^{\Delta s}, \ldots, \tilde{\pi}_{t,K}^{\Delta s})$, $\tilde{\boldsymbol{m}}_t^{\Delta s} = (\tilde{m}_{t,1}^{\Delta s}, \ldots, \tilde{m}_{t,K}^{\Delta s})$ and $\tilde{\boldsymbol{\sigma}}_t^{\Delta s} = (\tilde{\sigma}_{t,1}^{\Delta s}, \ldots, \tilde{\sigma}_{t,K}^{\Delta s})$. Thus, the swap rate S_{t+1}^n is also a mixture of normals under $\tilde{\mathbb{P}}$ with parameters $\tilde{\pi}_{t,k}^{s} = \tilde{\pi}_{t,k}^{\Delta s}$, $\tilde{\sigma}_{t,k}^{s} = \tilde{\sigma}_{t,k}^{\Delta s}$ and $\tilde{m}_{t,k}^{s} = S_t^n + \tilde{m}_{t,k}^{\Delta s}$ for $k = 1, \ldots, K$. At this point it is helpful to introduce the state variable V_t with $\tilde{P}_t(V_t = k) = \tilde{\pi}_{t,k}^{\Delta s}$ and $\Delta S_{t,1}^n | V_t = k \sim$

$N(\tilde{m}^{\Delta s}_{t,k}, (\tilde{\sigma}^{\Delta s}_{t,k})^2)$. It follows that

$$\tilde{\mathbb{E}}_t[(S^n_{t+1} - Y_S)^+] = \sum_{k=1}^{K} \tilde{\pi}^{\Delta s}_{t,k}\ \tilde{\mathbb{E}}_t[(S^n_t + \Delta S^n_{t,1} - Y_S)^+ | V_t = k], \tag{36}$$

$$\tilde{\mathbb{E}}_t[(Y_S - S^n_{t+1})^+] = \sum_{k=1}^{K} \tilde{\pi}^{\Delta s}_{t,k}\ \tilde{\mathbb{E}}_t[(Y_S - S^n_t - \Delta S^n_{t,1})^+ | V_t = k]. \tag{37}$$

With the Bachelier pricing formula,[65] it follows that $\tilde{\mathbb{E}}_t[(S^n_{t+1} - Y_S)^+]$ is equal to

$$\sum_{k=1}^{K} \tilde{\pi}^{\Delta s}_{t,k} \left[\tilde{\sigma}^s_{t,k}\, \varphi\left(\frac{S^n_t + \tilde{m}^{\Delta s}_{t,k} - Y_S}{\tilde{\sigma}^s_{t,k}} \right) + (S^n_t + \tilde{m}^{\Delta s}_{t,k} - Y_S)\, \Phi\left(\frac{S^n_t + \tilde{m}^{\Delta s}_{t,k} - Y_S}{\tilde{\sigma}^s_{t,k}} \right) \right] \tag{38}$$

and $\tilde{\mathbb{E}}_t[(Y_S - S^n_{t+1})^+]$ is equal to

$$\sum_{k=1}^{K} \tilde{\pi}^{\Delta s}_{t,k} \left[\tilde{\sigma}^s_{t,k}\, \varphi\left(\frac{S^n_t + \tilde{m}^{\Delta s}_{t,k} - Y_S}{\tilde{\sigma}^s_{t,k}} \right) + (Y_S - S^n_t - \tilde{m}^{\Delta s}_{t,k})\, \Phi\left(\frac{Y_S - S^n_t - \tilde{m}^{\Delta s}_{t,k}}{\tilde{\sigma}^s_{t,k}} \right) \right]. \tag{39}$$

Finally, we test the quality of these approximations. Therefore, we compare the approximated prices with exact prices based on Monte Carlo simulations. We report the results for one-into-ten year swaption contracts and for two selected parameter sets in Table B.1. Whereas the first set of parameters reflects a rather calm market environment, the second parameter set is characterized by a higher volatility in combination with a higher probability for the second state. We assume a flat term structure with an annually compounded rate of 2%, which implies a fair ten-year spot swap rate S^{10}_t of 2%. Swaption prices for several strikes are reported in terms normal implied volatilities. Differences between approximated prices and exact prices tend to be relatively small. For the first (second) parameter set, relative errors lie between -2.9% and 4.1% (-7.7% and 4.4%). We find similar results for many other tested specifications and conclude that the approximation quality tends to be reasonably well. In Section III of the Online Appendix, we perform a more holistic test of our calibration approach where these approximations are used.

[65] See e.g. Schachermayer and Teichmann (2008).

Table B.1: Accuracy of the Approximations

Panel A: Mixture Models

	$\tilde{\pi}_{t,1}^{\Delta s}$	$\tilde{\sigma}_{t,1}^{\Delta s}$	$\tilde{\sigma}_{t,2}^{\Delta s}$	$\tilde{m}_{t,1}^{\Delta s}$	$\tilde{m}_{t,2}^{\Delta s}$
Parameters I	0.9	0.005	0.010	0.000	0.005
Parameters II	0.8	0.005	0.015	0.000	-0.005

Panel B: Results

		Parameters I			Parameters II		
Type	Y_S	exact	approx.	rel. error	exact	approx.	rel. error
Receiver	0.00	66.6	65.3	-1.9	126.3	120.7	-4.4
Receiver	0.50	59.9	58.8	-1.9	116.3	110.6	-4.9
Receiver	1.00	54.8	53.8	-1.8	103.8	97.8	-5.7
Receiver	1.50	52.5	51.3	-2.2	92.5	86.2	-6.9
Receiver	1.75	51.8	50.3	-2.9	89.9	83.0	-7.7
Straddle	2.00	54.6	55.1	1.0	72.7	70.2	-3.3
Payer	2.25	58.9	61.4	4.1	57.7	60.2	4.4
Payer	2.50	60.7	62.8	3.5	60.9	63.2	3.7
Payer	3.00	67.4	69.3	2.8	70.3	72.5	3.2
Payer	3.50	76.5	78.3	2.5	81.9	84.2	2.8
Payer	4.00	83.9	85.8	2.3	92.0	94.5	2.6

This table reports two parameter sets for the mixture model (Panel A) and the corresponding results of the simulation study (Panel B) where exact Monte Carlo prices for one-into-ten year swaption contracts are compared with prices based on the approximation formula. Prices are reported in terms of normal implied volatilities which are measured in basis points. A flat term structure with an annually compounded rate of 2% is assumed. The strike Y_S and relative errors are in percent.

C Further Results

Table C.2: Interest Rate Risk – Further Results

Panel A: Upward Shocks – 5Y							
	avg	std	min	q25	med	q75	max
imp	2.35	0.47	1.25	2.12	2.48	2.67	3.21
Solvency II	1.36	0.54	1.00	1.00	1.00	1.57	2.82
Solvency II*	2.34	0.72	1.34	1.71	2.09	2.87	3.89
norm-month	1.91	0.16	1.50	1.82	1.92	2.04	2.11
EGARCH	1.62	0.30	1.18	1.38	1.58	1.78	2.50
Panel B: Downward Shocks (abs.) – 5Y							
	avg	std	min	q25	med	q75	max
imp	1.23	0.63	0.51	0.68	1.02	1.64	3.19
Solvency II	0.79	0.73	0.00	0.13	0.52	1.31	2.36
Solvency II*	1.39	0.64	0.50	0.82	1.17	1.85	2.76
norm-month	1.91	0.16	1.50	1.82	1.92	2.04	2.11
EGARCH	2.15	0.32	1.55	1.94	2.13	2.32	3.12

Panel C: Upward Shocks – 20Y							
	avg	std	min	q25	med	q75	max
imp	2.07	0.86	0.88	1.39	1.96	2.78	4.02
Solvency II	1.05	0.09	1.00	1.00	1.00	1.06	1.33
Solvency II*	1.56	0.35	0.89	1.25	1.52	1.90	2.16
norm-month	1.66	0.30	1.19	1.39	1.79	1.94	2.03
EGARCH	1.39	0.15	0.92	1.32	1.41	1.47	1.82
Panel D: Downward Shocks (abs.) – 20Y							
	avg	std	min	q25	med	q75	max
imp	1.34	0.74	0.11	0.69	1.36	1.95	2.87
Solvency II	0.79	0.41	0.02	0.43	0.74	1.18	1.48
Solvency II*	1.86	0.71	0.53	1.23	1.78	2.54	3.05
norm-month	1.66	0.30	1.19	1.39	1.79	1.94	2.03
EGARCH	2.25	0.22	1.80	2.07	2.27	2.38	3.01

This table reports summary statistics for upward and downward shocks to the 5-and 20-year spot rates calculated at the end of each month of our sample period between 2006 and 2019. The shocks are constructed to be in line with the 99.5%-VaR objective, i.e., they correspond to the 0.5%- and the 99.5%-quantiles of the change in the spot rate over one year. We report results for our option-implied approach relying on two-state mixtures and a quadratic (log-)SDF (imp), the current Solvency II standard formula (Solvency II) and the revised Solvency II methodology (Solvency II*). Furthermore, we include a Gaussian benchmark based on monthly data (norm-month) and we report results for simulation-based EGARCH quantile forecasts. For each methodology, we report the time-series average (avg), the standard deviation (std), the minimum (min), the 25%-quantile (q25), the median (med), the 75%-quantile (q75) and the maximum (max). Summary statistics for the downward shocks are in absolute values. Numbers are in percentage points.

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Online Appendix for
Option-Implied Solvency Capital Requirements

Abstract: This Online Appendix consists of five sections. Section I provides details for the derivation of the general relationship between risk-neutral and physical densities. Section II presents standard pricing results for swaptions which are used for the conversion between prices and implied volatilities. Section III describes a simulation study which examines the robustness of our calibration results for interest rate risk. In Section IV, we provide details on the implementation of our benchmark models. Finally, Section V contains further empirical results.

I Measure Change

In this section, we explain how equation (8) is obtained from the pricing results in equations (6) and (7). Assuming that $X_{t,\tau}$ has a density $\tilde{f}_{t,\tau}$ with respect to the forward risk-neutral measure $\tilde{\mathbb{P}}$, we obtain from equation (6)

$$p_{t,i} = B_{t,\tau} \int v_i(x)\, \tilde{f}_{t,\tau}(x)\, dx. \tag{I.1}$$

Similarly, equation (7) implies

$$p_{t,i} = \int m^*_{t,\tau}(x)\, v_i(x)\, f_{t,\tau}(x)\, dx \tag{I.2}$$

where $f_{t,\tau}$ denotes the density of $X_{t,\tau}$ with respect to the physical measure $\mathbb{P}$. These two pricing results hold for *any* payoff. Comparing the integrals in the equations (I.1) and (I.2), we then obtain

$$B_{t,\tau}\, \tilde{f}_{t,\tau}(x) = m^*_{t,\tau}(x)\, f_{t,\tau}(x) \tag{I.3}$$

and thus

$$f_{t,\tau}(x) = \frac{1}{m^*_{t,\tau}(x)} B_{t,\tau} \tilde{f}_{t,\tau}(x). \tag{I.4}$$

Due to the normalization of density functions, it holds that

$$1 = \int f_{t,\tau}(s)\, ds = B_{t,\tau} \int \frac{\tilde{f}_{t,\tau}(s)}{m^*_{t,\tau}(s)}\, ds. \tag{I.5}$$

Using this result, we can eliminate $B_{t,\tau}$ in equation (I.4) and obtain equation (8).

II Normal Pricing Formulas

In this section, we report the pricing formula for USD-swaptions which are typically used to convert between prices and implied (normal) volatilities also for EUR-swaptions. For the normal model, we get the following prices for τ-year into n-year payer and receiver swaptions:[66]

$$\pi_{PS}(t) = A_t^{\tau,n}\,[(\hat{s}_t^{\tau,n} - Y_S)\Phi(d) + \sigma_{norm}\sqrt{\tau}\,\varphi(d)] \quad \text{and} \tag{II.1}$$

$$\pi_{RS}(t) = A_t^{\tau,n}\,[(Y_S - \hat{s}_t^{\tau,n})\Phi(-d) + \sigma_{norm}\sqrt{\tau}\,\varphi(d)], \tag{II.2}$$

where φ is the pdf of the standard normal distribution and

$$d = \frac{\hat{s}_t^{\tau,n} - Y_S}{\sigma_{norm}\sqrt{\tau}}. \tag{II.3}$$

Furthermore, $A_t^{\tau,n} = \sum_{i=1}^{n} B_{t,\tau+i}$ where $B_{t,n}$ is the time-t price of a n-year zero-coupon bond with payoff 1 and

$$\hat{s}_t^{\tau,n} = \frac{B_{t,\tau} - B_{t,\tau+n}}{A_t^{\tau,n}}. \tag{II.4}$$

$\hat{s}_t^{\tau,n}$ is referred to as the forward swap rate, which is the fixed rate that implies a zero present value of the underlying swap at time t. At the maturity of the swaption, the forward swap rate is equal to the spot swap, that is, $\hat{s}_{t+\tau}^{\tau,n} = S_{t+\tau}^{n}$ (Trolle and Schwartz, 2014).

The underlying assumption in the normal model is that the forward swap rate evolves according to the stochastic differential equation (Milliman, 2015)

$$d\hat{S}_t^{\tau,n} = \sigma_{norm} dW_t, \tag{II.5}$$

where W_t is a standard Brownian motion under the specific risk neutral measure that uses $A_t^{\tau,n}$ as the numeraire ("annuity measure").[67] This implies that the spot swap rate $S_{t+\tau}^{n}$ at time $t+\tau$, is normally distributed with an expected value equal to the forward swap rate $\hat{s}_t^{\tau,n}$.

[66] See Milliman (2015).
[67] See e.g. Trolle and Schwartz (2014) for details.

III Simulation Study: Calibration

For interest rate risk, the risk-neutral two-state mixture models are determined by solving the minimization problem (15). For this calibration, there are typically only eleven swaption quotes available which is a much smaller number than for the equity case. Furthermore, we make use of the approximation formulas presented in Section B to avoid Monte Carlo pricing. For analyzing the accuracy and stability of this calibration procedure, we implement the following test:

(i) We start with a fully specified (risk-neutral) mixture model.

(ii) For this model, we calculate *exact* prices by Monte Carlo simulation of one-into-ten year swaptions for the eleven strikes that are typically available. We transform these prices into implied normal volatilities.

(iii) We apply our calibration procedure based on the implied volatilities from the previous step.

(iv) Ultimately, we compare the true probability distributions with the re-calibrated distributions.

The results of this calibration test are presented in Table III.1.[68] We observe comparable quantiles and moments which indicates similar risk-neutral distributions. For the 0.5%- and 99.5%-quantiles, which are particularly relevant for our tail risk assessments, errors range between −5.8% and 7.5% of the true quantiles. Overall, we think that these results support our calibration procedure for interest rate risk.

[68] Again, we assume a flat term structure of interest rates with an annually compounded rate of 2%. Thus, the fair ten-year spot swap rate S_t^{10} is also equal to 2% p.a.

Table III.1: Calibration Test

Panel A: Mixture Models

	$\tilde{\pi}^{\Delta s}_{t,1}$	$\tilde{\sigma}^{\Delta s}_{t,1}$	$\tilde{\sigma}^{\Delta s}_{t,2}$	$\tilde{m}^{\Delta s}_{t,1}$	$\tilde{m}^{\Delta s}_{t,2}$
(0)	0.90	0.005	0.0100	0.000	0.005
(1)	0.95	0.005	0.0100	0.000	0.005
(2)	0.60	0.005	0.0100	0.000	0.005
(3)	0.90	0.005	0.0050	0.000	0.005
(4)	0.90	0.005	0.0175	0.000	0.005
(5)	0.90	0.005	0.0100	0.000	-0.010
(6)	0.90	0.005	0.0100	0.000	0.010
(7)	0.80	0.005	0.0175	0.000	0.010
(8)	0.70	0.005	0.0175	0.000	0.010

Panel B: Risk-Neutral Quantiles and Moments

	Quantiles					Moments			
	0.5%	5%	50%	95%	99.5%	mean	std	skew	kurt
(0)	-1.41	-0.84	0.03	1.01	2.15	0.05	0.59	0.54	4.95
re	-1.44	-0.86	0.01	0.98	2.06	0.03	0.58	0.47	4.85
(1)	-1.35	-0.83	0.01	0.91	1.80	0.02	0.55	0.36	4.45
re	-1.37	-0.85	0.00	0.89	1.74	0.01	0.54	0.32	4.38
(2)	-1.75	-0.93	0.12	1.65	2.74	0.20	0.78	0.58	4.22
re	-1.81	-0.95	0.09	1.58	2.66	0.16	0.77	0.55	4.30
(3)	-1.27	-0.80	0.04	0.92	1.43	0.05	0.52	0.06	3.03
re	-1.29	-0.81	0.03	0.90	1.42	0.03	0.52	0.07	3.05
(4)	-2.38	-0.92	0.02	1.09	3.38	0.05	0.74	0.94	10.96
re	-2.56	-0.96	-0.01	1.03	3.18	0.02	0.73	0.64	10.81
(5)	-2.64	-1.19	-0.05	0.81	1.31	-0.10	0.64	-1.03	6.00
re	-2.74	-1.27	-0.07	0.80	1.30	-0.13	0.66	-1.07	6.01
(6)	-1.31	-0.81	0.05	1.19	2.64	0.10	0.64	1.03	6.00
re	-1.33	-0.83	0.03	1.13	2.55	0.07	0.63	0.97	5.93
(7)	-2.43	-0.95	0.06	2.18	4.43	0.20	0.99	1.51	8.73
re	-2.57	-0.98	0.03	1.93	4.21	0.14	0.94	1.36	9.08
(8)	-2.72	-1.04	0.11	2.69	4.72	0.30	1.14	1.25	6.73
re	-2.89	-1.09	0.06	2.45	4.49	0.22	1.09	1.13	7.13

The table reports parameter sets for the mixture models (Panel A) and the corresponding results of the calibration test (Panel B). We report quantiles and moments for the true and the re-calibrated distributions of the swap rate change $\Delta S^{10}_{t,1}$. In Panel B, quantiles, means and standard deviations are measured in percent.

IV Time-Series Benchmarks

We use the following benchmark estimators for the variance and for selected quantiles of the risk factor changes over the period $[t, t+\tau]$.

As a simple benchmark motivated by the SST (Finma, 2017), we consider rolling window forecasts based on monthly risk factor data. In particular, we estimate the variance over $[t, t+\tau]$ from 10 years of monthly data according to

$$\sigma^2_{t,\mathrm{mon}}(\tau) = \tau\, 12 \left(\frac{1}{120} \sum_{s=1}^{120} (x_{s,m} - \hat{\mu}_t)^2 \right), \tag{IV.1}$$

where $(x_{s,m})_{s=1,\ldots,120}$ are the most recent 120 risk factor changes realized before month t and $\hat{\mu}_t$ is the sample average of these data. Here, time-scaling is applied to transform the one-month variance to the time horizon τ.

As a second model-free variance estimator, we implement a realized variance forecast based on 252 daily returns, i.e., we estimate the variance for the period $[t, t+\tau]$ at the end of month t as

$$\sigma^2_{t,\mathrm{day}}(\tau) = \tau \sum_{s=1}^{252} x^2_{s,d}, \tag{IV.2}$$

where $(x_{s,d})_{s=1,\ldots,252}$ are the 252 most recent daily realizations of the risk factor changes at the end of month t.

Again motivated by the regulatory approach proposed in Finma (2017), we additionally assume that the change of a given risk factor is normally distributed with an expected value of zero. Then, we obtain the simple quantile approximations

$$Q^{\mathrm{mon}}_{t,p}[X_{t,\tau}] = \Phi^{-1}(p)\, \sigma_{t,\mathrm{mon}}(\tau) \qquad \text{and} \qquad Q^{\mathrm{day}}_{t,p}[X_{t,\tau}] = \Phi^{-1}(p)\, \sigma_{t,\mathrm{day}}(\tau) \tag{IV.3}$$

from (IV.1) and (IV.2), where $\Phi^{-1}(p)$ denotes the p-quantile of the standard normal distribution.

Moreover, we use GARCH-based variance and VaR-forecasts. In particular, we assume the following data generating processes for the daily risk factor changes

$$X_t = \mu + \varepsilon_t \qquad \text{with} \qquad \varepsilon_t = \sigma_t Z_t, \tag{IV.4}$$

$$\sigma_t^2 = \omega + \alpha\, \varepsilon_{t-1}^2 + \beta\, \sigma_{t-1}^2, \tag{IV.5}$$

where $Z_t \sim N(0,1)$. We also consider the asymmetric EGARCH-model proposed by Nelson (1991). The corresponding data generating process is given by (IV.4) and

$$\ln(\sigma_t^2) = w + \alpha\, \varepsilon_{t-1} + \gamma \left(|\varepsilon_{t-1}| - \sqrt{2/\pi} \right) + \beta\, \ln(\sigma_{t-1}^2). \tag{IV.6}$$

EGARCH-specifications capture the leverage effect which might be advantageous for modeling volatility dynamics in stock markets and they have also been applied in the context of interest rates (Li and Zhao, 2009). In all cases, we estimate the parameters with Maximum Likelihood and update them every month. If not stated otherwise, we use a growing estimation window.

To obtain multiperiod variance and quantile forecasts from the estimated models, we rely on Monte Carlo simulation. For each forecasting day t, we simulate 10 000 random paths with the relevant number of trading days. Then, we aggregate the daily risk factor changes into changes over the time horizon $[t, t+\tau]$ and estimate the variance or quantiles from the empirical distribution of the simulated sample. The forecasts for the horizon $[t, t+\tau]$ are denoted by $\sigma^2_{t,(E)GARCH}(\tau)$ and $Q^{(E)GARCH}_{t,p}[X_{t,\tau}]$. We use standard normal innovations for the simulations or resample the empirical innovations. The latter approach is known as Filtered Historical Simulation in the risk management literature (Barone-Adesi et al., 1998).

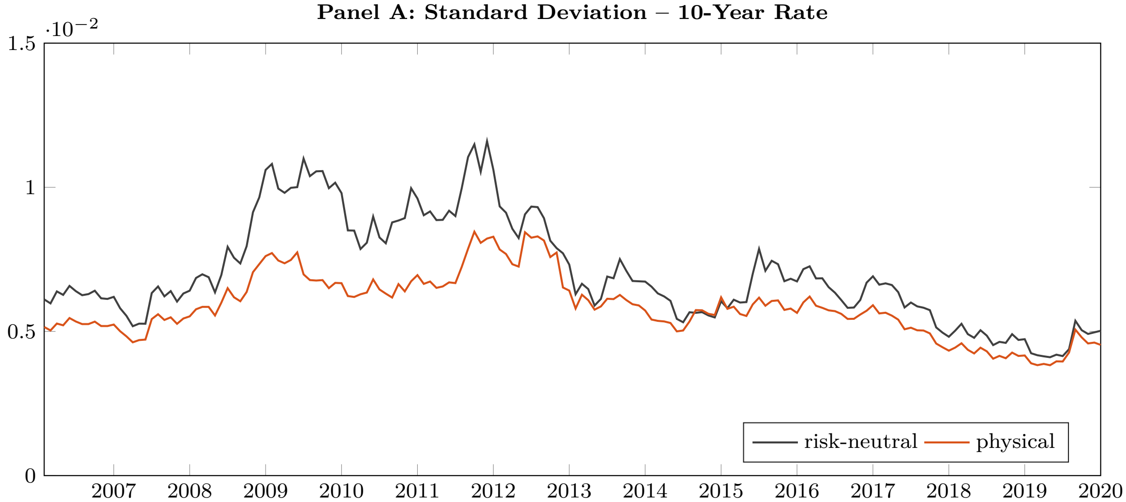
For the interest rate risk analysis, we use the Vasicek (1977) model as a further benchmark. In contrast to other short rate models, such as the CIR model, the Vasicek model allows for negative rates. In this model, the short rate is given by

$$dr_t = \alpha(\gamma - r_t)\, dt + \sigma\, dW_t \tag{IV.7}$$

with a long-term mean level of γ, speed of mean reversion α, volatility σ and (risk-neutral) Brownian motion (W_t). Following Gatzert and Martin (2012), we assume a zero market price of risk and also use three-month EURIBOR rates to estimate the model parameters by maximum likelihood estimation.[69] The physical quantiles of the one-year spot rate changes are then determined by Monte Carlo simulations, using closed-form expressions for zero bond prices (Björk, 2009, Prop. 24.3) to determine the corresponding n-year spot rates.

[69] A growing estimation window is used and we slightly restrict the mean parameter γ for stability reasons.

V Additional Empirical Results



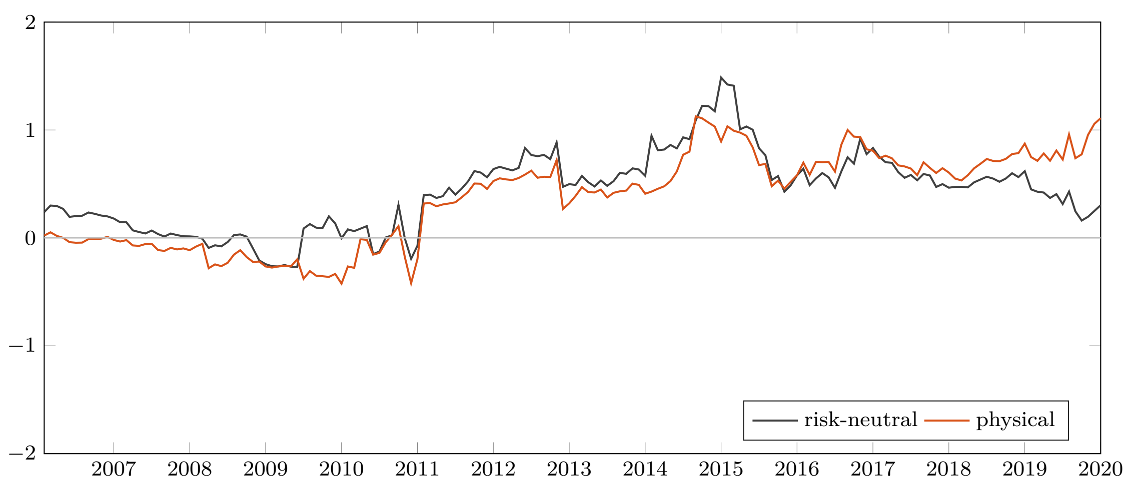


Figure V.1: The figure shows option-implied risk-neutral and physical moments of the 1-year change in the 10-year spot rate ($\Delta R^{10}_{t,1}$).

Table V.2: Interest Rate Risk (10Y) – Robustness and Risk Neutral Results

Panel A: Upward Shocks							
	avg	std	min	q25	med	q75	max
imp	2.01	0.57	0.97	1.67	2.10	2.47	2.97
imp-roll	2.00	0.57	0.97	1.67	2.07	2.40	3.01
imp-vol-error	2.04	0.57	0.98	1.69	2.12	2.52	3.02
imp-hist	1.86	0.48	0.96	1.56	1.88	2.25	2.80
imp-GARCH	1.85	0.62	0.70	1.39	1.99	2.38	2.84
imp-rn	2.38	0.54	1.44	1.94	2.39	2.69	3.96
imp-rn-1s	1.79	0.42	1.15	1.48	1.68	2.10	2.91
Panel B: Downward Shocks (abs.)							
	avg	std	min	q25	med	q75	max
imp	1.39	0.62	0.63	0.85	1.16	1.83	2.83
imp-roll	1.40	0.61	0.68	0.86	1.15	1.83	2.83
imp-vol-error	1.40	0.63	0.62	0.85	1.17	1.84	2.84
imp-hist	1.27	0.47	0.65	0.86	1.09	1.62	2.47
imp-GARCH	1.36	0.66	0.58	0.77	1.08	1.85	2.92
imp-rn	1.57	0.56	0.84	1.19	1.43	1.78	3.30
imp-rn-1s	1.79	0.42	1.15	1.48	1.68	2.10	2.91

This table reports descriptive statistics of the option-implied upward and downward shocks for the 10-year rate. The results of our baseline specification (imp) are compared to the modifications described in Figure 4. Additionally, option-implied risk-neutral results for 1-state mixtures (imp-rn-1s) and 2-state mixtures (imp-rn) are shown. We report the time-series average (avg), the standard deviation (std), the minimum (min), the 25%-quantile (q25), the median (med), the 75%-quantile (q75) and the maximum (max). Statistics for the downward shocks are reported in absolute values. Numbers are in percentage points.

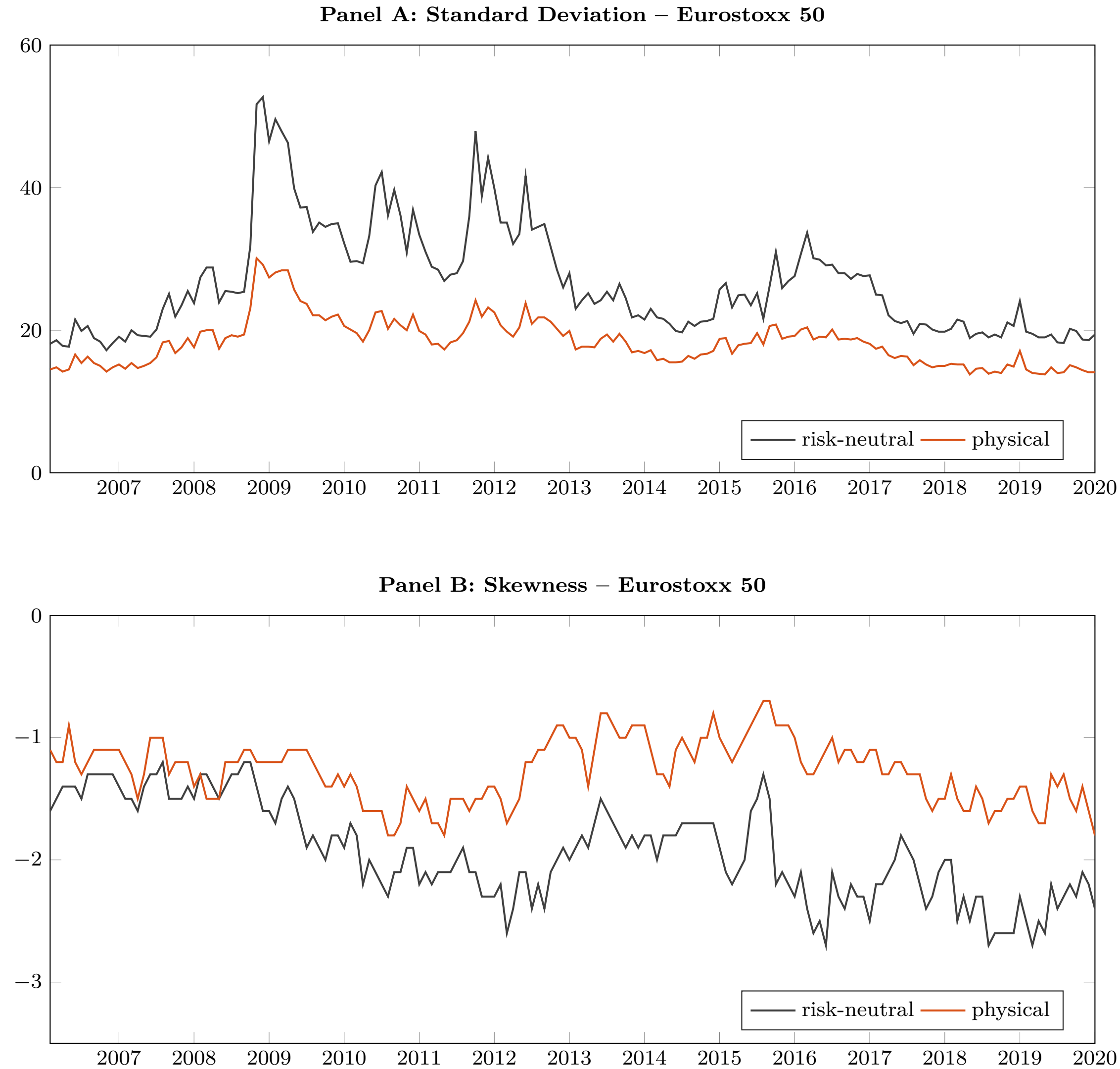


Figure V.2: The figure shows option-implied risk-neutral and physical moments of the 1-year log-return for the Eurostoxx 50. These moments are derived from the two option cross-sections using linear interpolation on the level of the variances and skewness coefficients.

Table V.3: Equity Risk – Robustness and Risk Neutral Results – Eurostoxx 50

	avg	std	min	q25	med	q75	max
imp	47.76	5.14	38.41	43.99	47.10	51.01	62.33
imp-2s	52.23	5.97	41.09	48.77	50.60	56.32	66.73
imp-4s	48.68	5.40	38.60	44.31	47.79	52.44	63.14
imp-5s	48.84	5.37	38.65	44.83	47.96	52.66	62.72
imp-roll	48.75	5.00	38.55	45.99	48.24	51.67	62.77
imp-vol-error	49.02	5.40	39.03	45.01	48.16	52.57	64.28
imp-mse	46.57	4.85	37.75	42.82	46.07	49.56	60.22
imp-hist	47.62	6.57	36.41	42.36	47.19	52.06	64.59
imp-GARCH	47.66	5.13	38.36	43.86	47.01	50.91	62.18
imp-quadratic	48.46	3.32	42.37	45.84	48.19	50.59	58.38
imp-m15	47.76	5.28	38.42	43.87	47.13	51.04	62.36
imp-rn	70.29	9.88	48.51	63.63	68.87	78.52	89.38
imp-rn-1s	50.23	5.99	38.10	44.91	50.56	54.81	65.49

This table reports summary statistics for the monthly time series of Eurostoxx 50 shocks from 2006 until 2019. The shocks are constructed to be in line with the 99.5%-VaR objective over a one-year horizon. The results of our baseline specification (imp) are compared to the modifications described in Figure 6. Additionally, option-implied risk-neutral results for 1-state mixtures (imp-rn-1s) and 3-state mixtures (imp-rn) are shown. For each series, we report the average (avg), the standard deviation (std), the minimum (min), the 25%-quantile (q25), the median (med), the 75%-quantile (q75) and the maximum (max). Numbers are in percentage points.

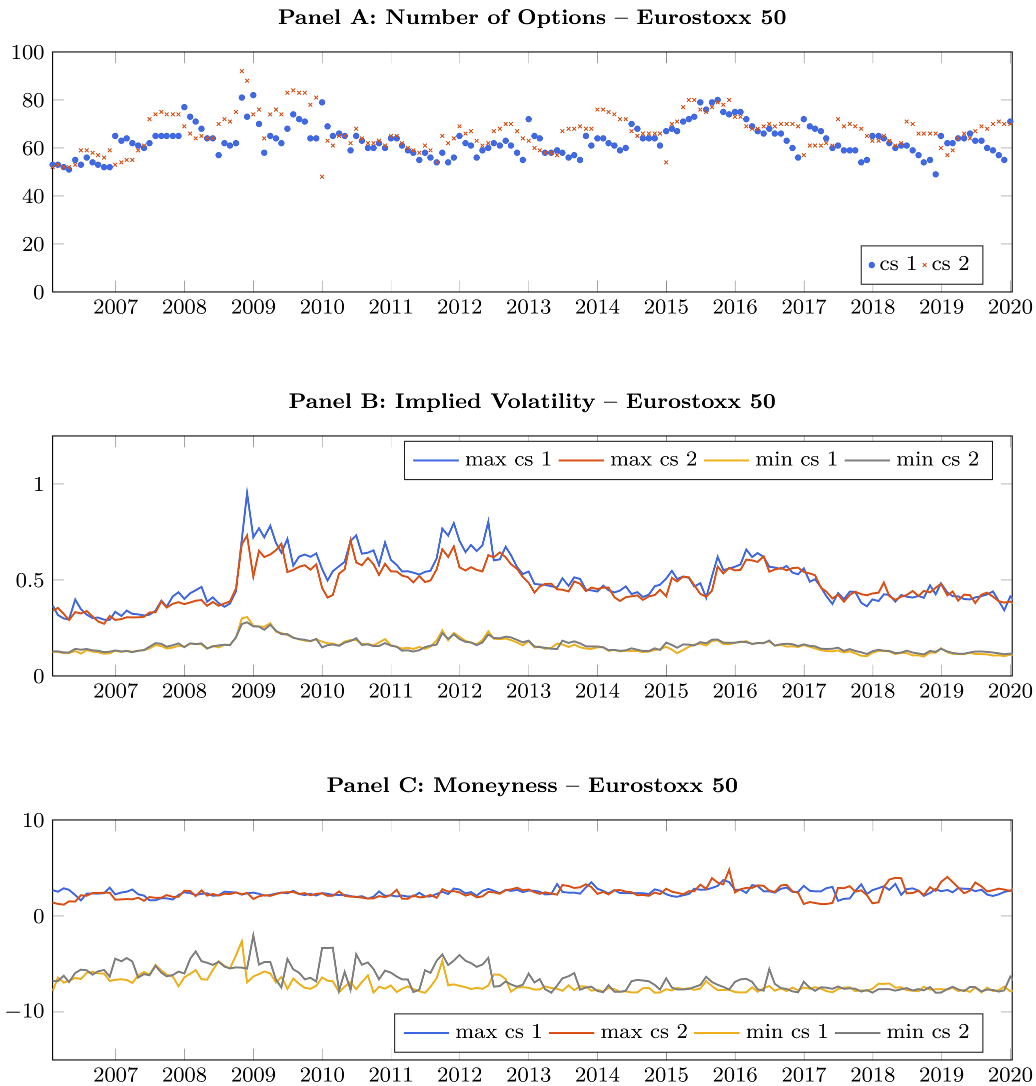


Figure V.3: The figure presents characteristics of the Eurostoxx 50 option data. Panel A shows the number of options per observation date that are used for the calibration of the risk neutral distributions. In Panel B, we plot the range of Black-Scholes implied volatilities in our option samples over time and Panel C shows the moneyness range, where moneyness is defined by formula (23). cs1 (cs2) refers to the first (second) cross section with $\tau < 1$ ($\tau > 1$).