Risk Pooling and Solvency Regulation: A Policyholder's Perspective*

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Abstract: We investigate the benefits of risk pooling for the policyholders of stock insurance companies under different solvency standards. Using second degree stochastic dominance, we document that the utility of risk-averse policyholders is increasing in the pool size if the equity capital is proportional to the premiums written. To the contrary, an increase in the pool size can reduce the policyholders' utility if the equity capital is determined using the Value-at-Risk (VaR). We show that pooling with a larger number of risks is also beneficial for all risk-averse policyholders under a VaR-based regulation if the pool satisfies an excess tail risk restriction. Our analysis provides new insights for the design of solvency standards and reveals a potential disadvantage of risk-based capital requirements for policyholders.

Keywords: Risk Pooling, Solvency Regulation, Value-at-Risk, Exchangeable Risks, Excess Wealth Order

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1 Introduction

Risk reduction through pooling can be seen as a defining characteristic of the insurance mechanism from the insurer's perspective. For example, Houston (1964, p. 538) argues that:

> "To the individual, insurance is a device for transferring his risk to the insurer. The insurer views insurance as a pooling process in which the risk is reduced by increasing the number of exposure units insured."

Following this view, the benefits of larger risk pools are typically studied by quantifying the reduction of the *insurer's risk* as measured by its default probability or the relative capital buffer (see, e.g., Cummins, 1974, 1991). However, if we take potential losses from a default into account, then the size of the insurance risk pool can also affect the payoffs to policyholders and thus their overall utility from the above-mentioned risk transfer.[1]

In this paper, we reinvestigate the benefits of risk pooling from the policyholders' perspective under different solvency frameworks. We focus on the case of a stock insurer, which is of special interest because the benefits of reducing the risk per policy have to be shared between the policyholders and the owners of the company. This risk allocation can drive a wedge between the occurrence of pooling benefits for the insurer's total position and the policyholders' wealth. Given the limited liability of equity holders, the default risk that the policyholders have to bear for a given amount of total risk depends on the equity capital that the owners of the company provide. We assume that this equity contribution is exogenously determined according to solvency rules and we consider two cases: minimum capital requirements that are proportional to the total premiums written and a capital regulation that is based on the Value-at-Risk (VaR). Capital requirements that are proportional to (net) premiums are an important example for volume-based systems such as the capital charges for underwriting risk in the United States and the former European Solvency I framework. The VaR-based rule has become a main component of probabilistic solvency systems around the world, for example, in the European Solvency II framework.[2]

[1] Several studies have argued that policyholders are highly sensitive to non-performance risks of insurance contracts. Cf. the discussion in Froot (2007) and the literature on "probabilistic insurance" (see, e.g., Wakker et al., 1997; Zimmer et al., 2018) for empirical results. This also applies to markets with guaranty funds whose protection is often only incomplete and associated with additional transaction costs (see Cummins and Sommer, 1996, p. 1075 or Cummins and Weiss, 2016, p. 130).

[2] Holzmüller (2009) and Cummins and Phillips (2009) provide reviews of the solvency regulation in the United States and in Europe. See Geneva Association (2016) for the results of a recent global survey.

Our baseline analysis relies on the following main assumptions: For the risks being insured, we only require homogeneity and finite expectations. Homogeneity is formalized by assuming that the random losses are exchangeable, which includes independent and identically distributed losses as a special case. The finiteness of expectations is necessary to evaluate the resulting wealth positions within an expected utility framework. We apply a second degree stochastic dominance criterion to obtain utility comparisons that are consistent with the preferences of risk-averse agents across a wide range of decision models. Default losses for policyholders are modeled endogenously by comparing the total claim amount to the level of the available reserves (equity capital and premiums) following ideas developed by Merton (1974) and Doherty and Garven (1986). Furthermore, we do not apply a specific pricing model but take the insurance premium as exogenously given. Finally, we assume that the policyholders are offered full coverage and that the total default loss from a given pool is shared equally among the policyholders.

These assumptions are sufficient to generate monotonically increasing benefits of risk pooling on the pool level. More specifically, the riskiness of the average claim per policyholder is non-increasing in the pool size under the given assumptions, in line with the general notion that diversification reduces risk. However, we demonstrate that the resulting effect on the policyholders' utility depends on the form of capital regulation due to the asymmetric risk sharing between policyholders and equity holders.

Under a simple volume-based solvency framework in which the equity capital is proportional to the premiums, the pooling benefits for policyholders are consistent with the overall risk reduction. In particular, we show that the policyholders' utility level is non-decreasing in the pool size so that all risk-averse policyholders at least weakly prefer insurance in larger risk pools.

In contrast, the occurrence of a risk reduction on the pool level does not necessarily translate into utility gains for policyholders if the amount of equity capital is determined using the VaR. Although, by construction, a VaR-based equity capital limits the probability of default, the relationship between the pool size and the policyholders' utility level depends on the distribution of the risks that are pooled. Varying the distributional assumption on the losses, we illustrate that the policyholders' utility level can be (i) globally increasing, (ii) locally decreasing or even (iii) globally decreasing in the size of the risk pool. We then derive a condition that is necessary and sufficient for non-negative pooling benefits under a VaR-based regulation. In particular, our results relate a preference for larger risk pools to a decrease in the excess tail risk of the average claim as measured by the difference between Average Value-at-Risk (AVaR) and VaR. In addition, we provide sufficient conditions on the joint distribution of individual risks, which imply that the excess tail risk condition for the average claim of the pool is satisfied.

Finally, we investigate a case in which the policyholders also own an equity stake in the insurance company. In this case, the effect of risk pooling on the policyholders' utility is always nonnegative – independent of the form of the minimum capital requirements.

We then discuss several extensions of our baseline analysis: First, we take a variable expense loading into account and confirm the intuition that cost benefits can reinforce risk-related pooling benefits or compensate pooling-related utility losses resulting from increases in excess tail risk. Moreover, we demonstrate that the benefits of risk pooling are robust to introducing independent investment risk if the equity capital is proportional to the premiums. To obtain a corresponding result under VaR-based capital requirements, we have to impose an additional shape restriction on the distribution of the investment return. We then study the special case of independent risks, which allows us to relax our full coverage and equal loss sharing assumptions and to derive sufficient conditions for utility gains from risk pooling with more general contract types and with alternative sharing rules for the total default loss. Next, we extend our analysis to heterogeneous risk pools and derive extensions of our results for risks with differences in expected losses and different levels of dispersion. Finally, we show that a risk-based premium which reflects the default risk of the company can at least partly resolve the adverse effects documented for VaR-based capital requirements and distributions that do not satisfy our excess tail risk condition.

Our analysis is related to the literature on the benefits of risk pooling and diversification. As mentioned above, several authors have studied the relationship between the size of the risk pool and the insurer's risk, often applying asymptotic arguments that build on the law of large numbers or the central limit theorem (see, Houston, 1964; Cummins, 1974, 1991; Smith and Kane, 1994 among others). However, the impact of risk pooling on the policyholders' utility has so far mainly been investigated in the mutual insurance case. In particular, Gatzert and Schmeiser (2012) and Albrecht and Huggenberger (2017) show that policyholders benefit from risk pooling in this setting by exploiting that mutual insurance companies attain a complete sharing of profits and losses independent of premiums or capital reserves. Due to this insight, the analysis of the mutual insurance case is related to general results on diversification benefits for risk-averse decision makers (Samuelson, 1967; Rothschild and Stiglitz, 1971).[3] For the case of stock insurers, we are only aware of the recent work by Schmeiser and Orozco-Garcia (2021) who compare pooling benefits in mutual and stock insurance companies focussing on conditions under which policyholders attain the same utility levels from both organizational forms.[4] We extend the previous literature by providing a general analysis of pooling benefits for policyholders in stock insurance companies. This analysis reveals that risk pooling can have a negative impact on the policyholder's utility under a VaR-based regulation.

Given this finding, our work complements a number of recent studies that highlight adverse effects of diversification. Ibragimov (2009) shows that diversification can increase the overall VaR when pooling risks with extremely heavy tails.[5] Ibragimov et al. (2009) document the occurrence of "diversification traps" in reinsurance markets, which are due to locally decreasing utility levels from diversification with, again, heavy tailed risks. Furthermore, Ibragimov et al. (2011) demonstrate that the optimal level of diversification from the perspective of financial intermediaries can have adverse welfare implications. An important common feature of Ibragimov et al. (2011) and our analysis is that limited liability can cause adverse diversification effects for a group stakeholders. In addition, Ibragimov et al. (2011) also emphasize the relevance of distributional characteristics for their findings: so called "diversification disasters" only occur with fat-tailed risks in their model.[6] In contrast, our excess tail risk condition can also be violated when pooling light-tailed risks.

Moreover, our work is related to the economic literature on the role of default risk for the optimal design and the pricing of insurance contracts. In particular, our analysis of the policyholder's utility under default risk complements the work of Doherty and Schlesinger (1990) and subsequent studies (Cummins and Mahul, 2003; Mahul and Wright, 2004, 2007) that document how well-known results on optimal insurance purchasing have to be modified if indemnity payments are subject to default

[3]Cf. also Eeckhoudt et al. (1993), who study the interaction between diversification and insurance from the perspective of risk-averse decision makers.

[4]See also Laux and Muermann (2010) or Braun et al. (2015) and the references therein for previous comparisons of mutual insurance and stock insurance companies along different dimensions.

[5]Ibragimov and Walden (2007) obtain similar results for random variables with bounded support that are generated by truncating heavy-tailed distributions. Note that the VaR-based diversification limits documented with this approach do not apply in a utility-based analysis (with concave utility functions).

[6]See also Ibragimov et al. (2015) for a comprehensive review of the effects of heavy-tailedness on diversification and the related literature.

risk.[7] In addition, our endogenous modeling of default losses is related to the contingent claim approach for pricing default risk (Doherty and Garven, 1986; Cummins, 1988).[8] Despite these similarities, the focus of our analysis is different: We study the impact of risk pooling on the policyholder's utility losses from default – taking insurance prices as well as the offered form of coverage as given.

Finally, our results add to the literature on the optimal design of solvency regulation. While risk-sensitive capital requirements for insurance companies have a number of important advantages compared to volume-based frameworks (Cummins et al., 1993; Holzmüller, 2009), our analysis highlights the potential problem of the risk-based approach that a reduction of the capital buffer per policy for larger portfolios can adversely affect the policyholder's utility. Moreover, our results contribute to the ongoing debate on the adequacy of VaR for setting capital requirements. On the one hand, VaR has been heavily criticized because it is not always subadditive (Artzner et al., 1999) and does not sufficiently reflect the risk of extreme losses.[9] On the other hand, Dhaene et al. (2008) demonstrate that strong subadditivity can be undesirable for the determination of capital requirements since it can increase the shortfall risk after a merger. Building on this insight, they characterize the subadditivity level of VaR as "efficient".[10] We add a stakeholder-specific perspective to this debate by demonstrating that VaR-based capital requirements can reduce or even eliminate diversification benefits for the policyholders of stock insurers. Interestingly, this effect is not related to a lack of subadditivity in our examples but rather in line with the criticism of subadditivity by Dhaene et al. (2008).

We proceed as follows: Section 2 introduces our main assumptions and our decision-theoretic approach for evaluating pooling benefits. In Section 3, we present our baseline results on the benefits of risk pooling under full coverage and an equal sharing of default losses. In Section 4, we investigate selected extensions of our baseline analysis. Section 5 concludes. All proofs are given in the Appendix. An Online Appendix presents complementary results.

[7] Recently, Reichel et al. (2021) investigate optimal insurance demand when policyholders are able to diversify default risk across several insurance companies.

[8] Furthermore, our methodology for sharing the default loss between policyholders is related to ideas that have been applied to the pricing of default risk in multiline insurance companies (Phillips et al., 1998; Myers and Read, 2001; Ibragimov et al., 2010).

[9] Cf., for example, Dowd and Blake (2006) for a comprehensive review of VaR and its alternatives with a focus on insurance applications.

[10] In addition, Kou and Peng (2016) recently show that VaR satisfies an alternative set of axioms and emphasize the robustness of VaR compared to risk measures that are more sensitive to extreme tail events.

2 Decision Framework

In this section, we present our baseline assumptions, the solvency standards that we consider and the methodology used for general utility comparisons.

2.1 Baseline Assumptions

We study a one-period model with n agents, $n \in \mathbb{N}$. Agent i possesses the deterministic initial wealth $w_{0,i}$ at time $t = 0$ and faces a single risk with the potential loss X_i at time $t = 1$, $i = 1, \ldots, n$. To simplify the exposition, we assume that the risk-free rate is zero.[11] Without insurance, the end-of-period wealth of agent i is then given by

$$W_i = w_{0,i} - X_i. \tag{1}$$

For the distribution of the losses, we mainly rely on the following assumption:

Assumption A1 *The losses $(X_1, \ldots, X_n)$ are exchangeable with $\mathbb{E}[|X_i|] < \infty$ for all $i = 1, \ldots, n$.*

Exchangeability means that $(X_1, \ldots, X_n) \overset{\mathrm{d}}{=} (X_{\Pi(1)}, \ldots, X_{\Pi(n)})$ for all permutations $(\Pi(1), \ldots, \Pi(n))$ of the indices $(1, \ldots, n)$ (McNeil et al., 2015, p. 234).[12] This implies that the marginal distributions of the losses are identical. Assumption A1 thus captures the notion of a homogeneous risk pool in a rather general way. In particular, it includes the important special case of identically distributed and independent risks (IID). Compared to this special case, exchangeability allows for positive dependence. With respect to the marginal distributions, Assumption A1 does only require that the risks are absolutely integrable, but it does not restrict our analysis to specific parametric models.[13]

In our baseline analysis, we focus on insurance policies that offer full coverage:[14]

Assumption A2 *Agent i can buy full coverage of her loss X_i for the risk premium $\pi > 0$, $i = 1, \ldots, n$.*

[11] This assumption can easily be relaxed and does not affect our main conclusions.

[12] Cf., e.g., Denuit and Vermandele (1998) as well as Albrecht and Huggenberger (2017) and the references therein for actuarial applications of exchangeable random variables.

[13] Usually, it holds that $X_i \geq 0$. However, we do not need this additional restriction for our general analysis.

[14] Alternative types of contracts will be studied in Section 4.3.

We take the risk premium π as given and only impose the weak restriction $\pi > 0$.[15] In addition, we assume that every agent is offered the contract at the same price, i.e. $\pi_i = \pi$, which is a natural assumption for identically distributed risks. Furthermore, Assumption A2 implies that the premium does not vary with the number of policies sold. This is an important simplification in our baseline analysis that allows us to focus on a single channel through which pooling affects the policyholder's utility. Furthermore, it allows us to compare the policyholder's position across different pool sizes without having to assume a specific functional form that determines how the premium depends on the size of the risk pool.[16]

If agent i is able to buy an insurance policy that is not subject to default risk, her end-of-period wealth at $t = 1$ satisfies[17]

$$W_i^s = w_{0,i} - \pi. \tag{2}$$

This wealth position does not depend on the size of the risk pool, which is a crucial difference to the mutual insurance case, where the size of the pool directly affects the distribution of the policyholders' wealth as a result of their profit participation (Gatzert and Schmeiser, 2012; Albrecht and Huggenberger, 2017).

However, the simple position in equation (2) neglects that the funds of the insurance company are usually limited and that it might not be able to cover all claims at $t = 1$. We refer to insurance policies that are subject to this kind of default risk as *vulnerable contracts* (Johnson and Stulz, 1987; Cummins and Mahul, 2003) and we let $D_{i,n}$ denote the default loss of policyholder i, who bought insurance from a company with a total portfolio of n policies.[18] The final wealth of policyholder i from buying a vulnerable contract is then given by

$$W_{i,n} = w_{0,i} - \pi - D_{i,n}. \tag{3}$$

Our approach for modeling the distribution of $D_{i,n}$ relies on the assumption that all contracts

[15] We thus avoid restricting the validity of our results to specific pricing rules. Note that while $\pi > 0$ is sufficient for our formal analysis, tighter restrictions on the premium can usually be derived based on economic arguments. Such economic restrictions on the premium will be discussed in Section 3.5.

[16] As important generalizations of this baseline assumption, we will investigate variable expense loadings that depend on the size of the risk pool and an endogenous premium that varies with the default risk of the insurance company in Sections 4.1 and 4.5.

[17] Accordingly, we do not consider additional sources of risk, such as random initial wealth of the decision maker (Doherty and Schlesinger, 1983).

[18] Here, n is the total size of the risk pool including the contract of policyholder i.

are offered by a stock insurance company with limited liability. If the company sells n policies, its total claim amount can be calculated as

$$S_n = \sum_{i=1}^{n} X_i. \tag{4}$$

The funds available to cover these claims are the premium payments made by the policyholders and the equity capital that the owners of the insurance company provide at $t = 0$. The premiums correspond to $n\,\pi$ and the total equity available for a portfolio of size n is denoted by c_n. Due to the limited liability of the owners, default occurs if and only if

$$S_n > c_n + n\,\pi. \tag{5}$$

In this setting, the default probability for a pool of size n satisfies

$$PD_n = \mathbb{P}[S_n > c_n + n\,\pi] \tag{6}$$

and the total default loss is given by

$$L_n = \max(S_n - c_n - n\,\pi,\, 0). \tag{7}$$

L_n is the amount missing for fully covering all claims from the contracts in the pool. We thus work with default losses that are *endogenously* caused by high claim amounts.[19] $-L_n = -\max(S_n - c_n - n\,\pi,\, 0)$ corresponds to the payoff of the well-known default option, which the policyholders implicitly sell to the owners of the company.[20]

For most of our analysis, we assume that policyholders and owners are distinct groups.[21] To fully describe the impact of default on the *individual* policyholder's wealth, it then remains to define a rule for sharing the total default loss L_n from equation (7) among the policyholders.[22] A natural

[19]In Section 4.2, we introduce investment risk as an additional source of default losses.

[20]The idea of using contingent claim pricing to analyze corporate debt goes back to Merton (1974). In an insurance context, it has been introduced by Doherty and Garven (1986) and Cummins (1988).

[21]A combined policyholder and owner position will be investigated in Section 3.4.

[22]By dividing the *total* excess loss among the policyholders, we implicitly assume that the policyholders' claims are not protected by a guaranty fund. In Section 4.3, we will introduce a more general modeling of individual default losses that allows for partial compensation by a guaranty fund.

first choice is an equal distribution of L_n.[23] In this case, every policyholder's wealth is reduced by

$$\bar{L}_n := \frac{1}{n} L_n. \tag{8}$$

In our setting with identically distributed losses, this corresponds to an "ex ante" sharing rule[24], which splits the total default loss according to the policyholders' contribution to the total expected loss. If we assume that the policyholders are homogeneous with respect to their initial wealth and their risk preferences, an equal allocation of the default loss among policyholders can furthermore be shown to be the linear allocation with the highest aggregate expected utility across all policyholders.[25]

If we denote the average claim per policyholder by $\bar{S}_n := S_n/n$ and the available equity capital per policyholder by $\bar{c}_n := c_n/n$, then equation (8) can be rewritten as $\bar{L}_n = \max(\bar{S}_n - \bar{c}_n - \pi,\, 0)$.

The formal implications of the previous discussion are summarized in the following assumption:

Assumption A3 *The default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = \bar{L}_n$, $i = 1, \ldots, n$.*

Under the Assumptions A2 and A3, we can rewrite the policyholder's wealth from buying a vulnerable insurance contract as

$$W_{i,n} = w_{0,i} - \pi - \bar{L}_n = w_{0,i} - \pi - \max(\bar{S}_n - \bar{c}_n - \pi,\, 0). \tag{9}$$

While premiums and the available equity capital are substitutes with respect to controlling the default probability and the severity of default losses as can be seen from equations (6) and (7), equation (9) shows that this symmetry is lost when analyzing the policyholder's wealth. Furthermore, equation (9) reveals that the size of the risk pool affects the policyholder's wealth under the given assumptions through two channels: the distribution of $\bar{S}_n$ and the available equity capital per policyholder $\bar{c}_n$. The dependence of the policyholder's wealth on the size of the risk pool is an important difference to the case without default risk as it potentially generates benefits of risk

[23] See, for example, Gatzert and Schmeiser (2012, p. 191) for the equal sharing of the excess loss.

[24] Cf. Ibragimov et al. (2010) for a discussion of "ex ante" vs. "ex post" sharing rules. We will consider more general sharing rules in Section 4.3.

[25] We provide a precise formal statement and a derivation of this risk sharing result as Proposition II.1 in Section II of the Online Appendix.

pooling from the policyholders' perspective beyond the mutual insurance case.

2.2 Capital Regulation

To understand the effect of different solvency rules, we assume that the equity capital provided by the owners exactly corresponds to the minimum capital requirement for a given pool size. In particular, we investigate the benefits of risk pooling under the following two solvency rules.

First, we consider the case in which the risk capital c_n is proportional to the total premiums written $n\,\pi$, which captures volume-based solvency frameworks like the underwriting risk charges according to the RBC standards in the United States or the former European Solvency I rules (Cummins and Phillips, 2009; Holzmüller, 2009). Since we assume a homogeneous pool with an identical risk premium for all contracts, this type of regulation implies that the minimum equity capital increases proportionally with the size of the risk pool n, i.e.,

$$c_n = n\,c. \tag{10}$$

Equity holders then provide the same amount of equity capital $\bar{c}_n = c$ for each contract.

Second, we consider minimum capital requirements that are based on the Value-at-Risk (VaR). VaR-based capital requirements are a very important component of modern probabilistic solvency standards in many insurance markets around the world (Geneva Association, 2016, Table 1), such as the current European Solvency II framework. We define the VaR of the random loss L at the probability level α as

$$\mathrm{VaR}_\alpha[L] = Q_{1-\alpha}[L] \tag{11}$$

with $Q_u[X]$ denoting the u-quantile of the random variable X, i.e., $Q_u[X] = \inf\{x \in \mathbb{R}; \mathbb{P}[X > x] \leq 1 - u\}$. Under a VaR_α-based capital regulation, the minimum equity capital is calculated as

$$c_n = \mathrm{VaR}_\alpha[S_n - \pi\,n]\,. \tag{12}$$

By the definition of the VaR_α, this choice of the equity capital ensures that the insurer's default probability does not exceed the probability level α. Note that equation (12) can be rewritten as

$$\bar{c}_n = \mathrm{VaR}_\alpha\left[\bar{S}_n - \pi\right], \tag{13}$$

which typically implies that the equity capital per policyholder varies with the size of the risk pool. We illustrate this dependence for the simple case of normally distributed risks in the following example.

Example 1 *Suppose that the losses* $(X_1, \ldots, X_n)$ *are independent and normally distributed with* $X_i \sim \mathcal{N}(\mu, \sigma^2)$ *for all* $i = 1, \ldots, n$*, where* $\sigma > 0$*. Under Assumption A2, we obtain* $S_n \sim \mathcal{N}(n\mu, n\sigma^2)$ *for the total claim amount and* $\bar{S}_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\sigma^2\right)$ *for the average claim amount. Therefore, the available equity capital per policyholder according to equation* (13) *is given by*

$$\bar{c}_n = Q_{1-\alpha}[\bar{S}_n] - \pi = \mu - \pi + \Phi^{-1}(1-\alpha)\,\frac{1}{\sqrt{n}}\sigma \tag{14}$$

with $\Phi^{-1}(1-\alpha)$ *denoting the* $(1-\alpha)$*-quantile of the standard normal distribution. For* $\alpha < 0.5$*, it holds that* $\Phi^{-1}(1-\alpha) > 0$ *and the equity capital per policyholder is decreasing in the pool size* n*.*

The negative relationship between the amount of risk capital per policyholder that is required to maintain a given safety level as measured by the default probability is sometimes interpreted as a "benefit of risk pooling".[26] However, in the first place, this benefit only applies from the equity holders' perspective. Whether this situation is also advantageous for policyholders requires further investigation.

2.3 Stochastic Orders and Preferences

To study the impact of the pool size n on $W_{i,n}$ according to equation (3), we rely on the theory of decisions under risk. Choosing this framework, we implicitly assume that the policyholders cannot replicate their wealth positions using tradable assets. If a complete replication of the corresponding positions was possible, their value would correspond to the price of the respective replicating portfolio. However, it seems reasonable that a typical policyholder does not have access to instruments which can be used to replicate cashflows depending on the individual losses $X_1, \ldots, X_n$.

[26] See e.g. Gatzert and Schmeiser (2012), who refer to this effect as "case A" risk pooling.

More specifically, we use second degree stochastic dominance (SSD) for utility comparisons that are largely independent of a particular preference specification. Let W_1 and W_2 denote random wealth positions. W_1 is said to dominate W_2 by SSD ($W_1 \succeq_{ssd} W_2$) if $\mathbb{E}[\psi(W_1)] \geq \mathbb{E}[\psi(W_2)]$ for all non-decreasing concave functions ψ such that the expectations exist.[27] In the standard expected utility theory (EUT), this definition directly implies that all risk-averse agents with a utility function u_i satisfying $u_i' > 0$ and $u_i'' < 0$ weakly prefer W_1 over W_2 if $W_1 \succeq_{ssd} W_2$.

Given the limited ability of EUT to explain the findings of the empirical literature on "probabilistic insurance" (Wakker et al., 1997), it is important to note that our SSD analysis is also consistent with decision theories that incorporate probability weighting. In particular, SSD results reflect the preferences of risk-averse agents in the rank-dependent expected utility model (Quiggin, 1982; Yaari, 1987) if the utility function and the probability weighting function are concave. Formally, we assume that agent i assigns the value

$$V_i(W) = \int u_i(w)\, d(g_i \circ F_W)(w) \tag{15}$$

to W, where u_i denotes the agent's utility function with $u_i' > 0$ and $u_i'' < 0$, F_W is the cumulative distribution function of W and g_i is an agent-specific probability weighting function with $g_i(0) = 0$, $g_i(1) = 1$, $g_i' > 0$ and $g_i'' < 0$.[28] Then, it follows from Chew et al. (1987) that

$$W_1 \succeq_{ssd} W_2 \qquad \Rightarrow \qquad V_i(W_1) \geq V_i(W_2)\,. \tag{16}$$

Further relevant properties of SSD that will be used for analyzing the policyholders' utility are summarized in the Appendix.

[27] Cf. Hadar and Russell (1969) as well as Rothschild and Stiglitz (1970). Our analysis relies on the comprehensive discussion in Shaked and Shanthikumar (2007, Chapter 4), where SSD is introduced as increasing concave order. See also Levy (1992) for an extensive review of economic applications.

[28] Note that this value functional is related to the more general Choquet Expected Utility framework following Schmeidler (1989). In this framework, our assumption of a concave probability weighting function corresponds to a convex capacity. A representation of equations (15) and (16) in terms of Choquet Expected Utility can be found in Albrecht and Huggenberger (2017).

3 Baseline Results

In this section, we investigate the benefits of risk pooling under the baseline assumptions introduced in the previous section. First, we briefly describe the impact of risk pooling on the average risk of an insurance portfolio. Then, we present our main results on the policyholders' utility for a proportional growth of the available equity capital in Section 3.2 and for a Value-at-Risk-based rule in Section 3.3. In these analyses, we assume that policyholders and equity holders are distinct groups. In Section 3.4, we study the potential benefits of larger risk pools if the policyholders also provide the equity capital. A discussion of our results is finally provided in Section 3.5.

3.1 The Average Risk of the Pool

Under our baseline assumptions, the intuition that pooling reduces the risk per policyholder can be formalized as follows.

Lemma 1 *Suppose that the losses* $(X_1, \ldots, X_{n+1})$, $n \in \mathbb{N}$, *satisfy Assumption A1 and let* $\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ *denote the average loss of* $(X_1, \ldots, X_n)$, *then*

$$-\bar{S}_{n+1} \succeq_{ssd} -\bar{S}_n \qquad \textit{for all } n \geq 1. \tag{17}$$

This result directly follows from applying Lemma 3.1 in Albrecht and Huggenberger (2017) to $(-X_1, \ldots, -X_{n+1})$. Moreover, Lemma 1 is closely related to results in majorization theory.[29] Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we say that $\boldsymbol{a}$ is majorized by $\boldsymbol{b}$ and write $\boldsymbol{a} \prec \boldsymbol{b}$ if $\sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} b_{[i]}$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}$, where $x_{[i]}$ denotes the i-th largest element of $\boldsymbol{x}$ (Marshall et al., 2011, Definition 1.A.1). Intuitively, $\boldsymbol{a} \prec \boldsymbol{b}$ formalizes that $\boldsymbol{a}$ is "less spread out" than $\boldsymbol{b}$ (Marshall et al., 2011, p. 4). Building on this terminology, Lemma 1 can be derived from the relation

$$\left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, \frac{1}{n+1}\right) \prec \left(\frac{1}{n}, \ldots, \frac{1}{n}, 0\right) \tag{18}$$

and results from majorization theory as shown in Section II of our Online Appendix.

Economically, equation (17) states that all risk-averse decision makers prefer the average loss

[29] See Marshall et al. (2011) for a comprehensive review of this theory. We are grateful to an anonymous referee for bringing the interesting relation to this literature to our attention.

from a larger pool to the average loss from a smaller pool. Similar results, in particular for independent and identically distributed risks, are frequently exploited in the literature on risk pooling and diversification and can (at least) be traced back to Samuelson (1967) and Rothschild and Stiglitz (1971). Please note that the moment condition $\mathbb{E}[|X_i|] < \infty$ from Assumption A1 is important for the validity of such results on diversification benefits because it rules out risks with extremely fat tails, for which diversification is known to be ineffective or even to increase the risk of a position (Fama, 1965; Ibragimov, 2009).

To simplify the following presentation, it is helpful to rewrite the result of Lemma 1 in terms of increasing convex order (Shaked and Shanthikumar, 2007, Chapter 4), which is often referred to as stop-loss order in the actuarial literature (Denuit et al., 2005, p. 152). X is said to be smaller than Y in increasing convex order ($X \preceq_{icx} Y$) if $\mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)]$ for all non-decreasing convex functions ψ such that the expectations exist. With this definition, equation (17) is equivalent to[30]

$$\bar{S}_{n+1} \preceq_{icx} \bar{S}_n \qquad\qquad \text{for all } n \geq 1. \tag{19}$$

Accordingly the *average* risk of the pool is decreasing in the pool size if "risk" is measured in terms of increasing convex order.[31]

In the remaining part of this section, we investigate whether the policyholders whose contracts are pooled can benefit from this risk reduction on the pool level.

3.2 Volume-Based Capital Requirements

We first focus on a volume-based regulation with a constant amount of equity capital per policy. In this case, the default loss per policyholder is given by

$$\bar{L}_n^c = \max(\bar{S}_n - c - \pi,\, 0) \tag{20}$$

and we are able to derive the following result.

Proposition 1 *Suppose that the Assumptions A1, A2 and A3 hold. If the equity capital per policyholder is constant $\bar{c}_n = c$, then the utility benefits of risk pooling under default risk are increasing*

[30]The equivalence follows from property i) of Lemma 2 in the Appendix.

[31]We use "decreasing" for "non-increasing" and "increasing" for "non-decreasing" throughout this article.

in the pool size, i.e., $W_{i,n+1} \succeq_{ssd} W_{i,n}$ *for all* $n \geq 1$.

Proof: See the Appendix.

Proposition 1 states that risk pooling with a larger number of homogeneous risks is always more beneficial from the policyholders' perspective. As a consequence[32] of equation (16), all risk-averse agents with preferences according to equation (15) weakly prefer a larger degree of risk pooling, i.e.,[33]

$$V_i(W_{i,n+1}) \geq V_i(W_{i,n}) \qquad \text{for all } n \geq 1. \tag{21}$$

The benefits for policyholders are thus consistent with the results stated for the average risk of the pool at the beginning of this section.

The utility gains from insurance with larger risk pools originate from a decrease in the disutility from default states. In particular, the proof of Proposition 1 shows that

$$V_i\Big(-\bar{L}^c_{n+1}\Big) \geq V_i\Big(-\bar{L}^c_n\Big) \qquad \text{for all } n \geq 1. \tag{22}$$

When drawing economic conclusions from these utility comparisons, it is important to take the underlying assumptions into account, which focus our analysis on diversification effects. Of course, deviations from these assumptions, such as differences in the premiums charged, can offset the diversification benefits that occur in larger pools according to Proposition 1.

Similar to Lemma 1, Proposition 1 is closely related to majorization theory. In particular, an alternative proof of the proposition is again presented in Section II of our Online Appendix. In the Online Appendix, we also complement our results for finite n by an asymptotic analysis within the standard expected utility framework assuming independent risks. The results of this analysis show that the utility losses from default asymptotically vanish as $n \to \infty$ if the reserves per policyholder $(\pi + c)$ cover the expected claim amount (see Proposition I.1).

We now illustrate the result of Proposition 1 for selected preference specifications[34] and distributional assumptions.[35]

[32]This extension from SSD statements to preference relations according to the rank-dependent expected utility model based on equation (16) equally applies to our following results, but we will confine the remaining discussion to SSD statements for the sake of brevity.

[33]Note that establishing a *strict* SSD relation is not possible as explained in the proof of Proposition 1.

[34]Our examples rely on the standard expected utility framework that is obtained from equation (15) with $g_i(p) = p$.

[35]More detailed derivations for the presented examples can be found in the Online Appendix.

Example 2 *Throughout this example, we assume $w_{0,i} = 10$ for the policyholder's initial wealth and $c = 1$ for the available equity capital per policyholder. We rely on loss distributions with $\mathbb{E}[X_i] = 2$ and use $\pi = 2$ so that the premium exactly covers the expected claim.*

i) We first consider agents, whose risk preferences are given by the exponential utility function $u_i(w) = 1 - \exp(-\gamma_i \cdot w)$ with risk aversion parameter $\gamma_i > 0$. According to equation (9), the expected utility from buying a vulnerable contract can then be written as

$$\mathbb{E}[u_i(W_{i,n})] = \mathbb{E}\Big[1 - \exp\Big(-\gamma_i\,(w_{0,i} - \pi - \bar{L}^c_n)\Big)\Big] = 1 - \exp\left(-\gamma_i\,(w_{0,i} - \pi)\right)\,\mathbb{M}_{\bar{L}^c_n}(\gamma_i), \qquad (23)$$

where $\mathbb{M}_{\bar{L}^c_n}$ is the moment generating function of $\bar{L}^c_n$. Furthermore, we again assume normally distributed and independent risks $X_i \sim \mathcal{N}(\mu, \sigma^2)$, $i = 1, \ldots, n$. For this case, we know from Example 1 that $\bar{S}_n \sim \mathcal{N}(\mu, \frac{1}{n}\sigma^2)$. In the Online Appendix, we use this implication in combination with results on the partial moment generating function of the normal distribution to derive a closed-form expression for $\mathbb{M}_{\bar{L}^c_n}(\gamma_i)$. Furthermore, we show that $\mathbb{M}_{\bar{L}^c_n}(\gamma_i)$ is a decreasing function of n, which confirms that $\mathbb{E}[u_i(W_{i,n})]$ is increasing in the pool size. We illustrate the monotonic increase of the corresponding certainty equivalents in Panel A of Figure 1 for $\gamma_i = 0.5$, $\mu = 2$ and $\sigma = 4$. Furthermore, we include the certainty equivalent of buying a safe insurance policy and document that the certainty equivalent of $W_{i,n}$ converges to the level of the safe contract.[36]

ii) For the second part of our example, we maintain the preference specification from part i) but we assume that the risks are independent and follow a Gamma distribution $X_i \sim \mathcal{G}(a, b)$ with shape parameter a and scale parameter b. This implies that $S_n \sim \mathcal{G}(n\,a, b)$ and $\bar{S}_n \sim \mathcal{G}\Big(n\,a, \frac{b}{n}\Big)$. If $\gamma_i < \frac{1}{b}$, the partial moment generating function of the Gamma distribution exists for all n and we can rely on results derived in the Online Appendix to evaluate the expected utility according to equation (23). We choose $a = 4/3$ and $b = 3/2$ for our illustration. These parameters imply a lower standard deviation than in the Gaussian example but generate positive levels of skewness and excess kurtosis for the distribution of the individual losses. The resulting relationship between n and the certainty equivalent of buying a vulnerable contract is shown in

[36] See Example I.1 in the Online Appendix for formal results on the limit behavior of the utility losses from default under the given assumptions.

Panel B of Figure 1.

iii) *Next, we assume that the distribution of X_i is a normal mixture with two components capturing a low and a high loss state. Formally, we consider*

$$\mathbb{P}[X_i \leq x] = \mathbb{P}[Y_i = L]\ \Phi(x; \mu_L, \sigma_L^2) + \mathbb{P}[Y_i = H]\ \Phi(x; \mu_H, \sigma_H^2), \tag{24}$$

where μ_L (μ_H) and σ_L^2 (σ_H^2) denote the location and variance parameters in the low (high) loss state and $\Phi(\,\cdot;\, \mu, \sigma^2)$ denotes the cumulative distribution function of a normal distribution with mean μ and variance σ^2. Y_i is the state indicator for X_i with $\mathbb{P}[Y_i = L] = p_L > 0$, $\mathbb{P}[Y_i = H] = p_H > 0$ for all $i = 1, \ldots, n$ and $p_L + p_H = 1$. We assume that the state indicators $(Y_1, \ldots, Y_n)$ are independent. Then, the losses $(X_1, \ldots, X_n)$ are also independent and $\bar{S}_n$ follows a normal mixture distribution with $n + 1$ components as shown in the Online Appendix, where we also derive a representation of $\mathbb{M}_{\bar{L}_n^c}(\gamma_i)$. Using again equation (23) with $\gamma_i = 0.5$, we illustrate the resulting relationship between the pool size and the certainty equivalent of buying a vulnerable contract in Panel C of Figure 1 for $\mu_L = 0$, $\mu_H = 8$, $\sigma_L = \sigma_H = 0.1$, $p_L = \frac{3}{4}$ and $p_H = \frac{1}{4}$.[37]

iv) *We now modify the mixture structure presented in the previous part of this example such that it allows for a* common *crash scenario for all risks in the pool. In particular, we maintain the form of the marginal distributions shown in equation (24) for the individual losses but we assume that the state indicators are perfectly dependent, i.e, $\mathbb{P}[Y_i = Y_j] = 1$ for all $i, j = 1, \ldots, n$. The random variables $Y_1, \ldots, Y_n$ can thus be replaced by a single state variable, which is referred to as common mixture modeling in the actuarial literature (Wang, 1998). $\bar{S}_n$ then follows a two-state mixture and our results in the Online Appendix can again be used to evaluate the expected utility with an exponential utility function. In Panel D of Figure 1, we illustrate the resulting relationship between the pool size and the corresponding certainty equivalent for $p_L = 0.96$, $p_H = 0.04$, $\mu_L = 5/3$, $\mu_H = 10$, $\sigma_L = 4$ and $\sigma_H = 1$.*

v) *Finally, we consider an example with fat-tailed risks from a multivariate t distribution. More specifically, we assume $(X_1, \ldots, X_n)' \sim \mathcal{T}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ with $\mu_1 = \ldots = \mu_n = \mu$, $(\boldsymbol{\Sigma})_{ij} = 0$ for*

[37] These assumptions imply $\mathbb{E}[X_i] = 2$ and $\sigma[X_i] = 3.47$, so that the first two moments are roughly comparable to the normally distributed losses that we analyzed before. However, the mixture distribution is asymmetric with a skewness coefficient of 1.15.

$i \neq j$ and $(\mathbf{\Sigma})_{ii} = \sigma^2$, $i = 1, \ldots, n$. Then, it follows from the transformation properties of elliptical random vectors that $\bar{S}_n \sim \mathcal{T}(\mu, \frac{1}{n}\sigma^2, \nu)$. Since the moment generating function of the t distribution does not exist, we cannot use an exponential utility function in this case. Instead, we rely on mean-variance preferences and use the corresponding certainty equivalent wealth level

$$\mathrm{CEQ}^{mv}(W_{i,n}) = \mathbb{E}[W_{i,n}] - \frac{\gamma_i}{2}\,\mathrm{var}[W_{i,n}] \tag{25}$$

to assess the pooling benefits for policyholders.[38] *Under the given assumptions, $\mathrm{CEQ}^{mv}(W_{i,n})$ can be computed analytically as shown in the Online Appendix. Panel E of Figure 1 illustrates the relation between the pool size n and $\mathrm{CEQ}^{mv}(W_{i,n})$ for $\gamma_i = 1$, $\mu = 2$, $\sigma = 4/\sqrt{2}$, and $\nu = 4$. These values imply $\mathbb{E}[X_i] = 2$ and $\sigma[X_i] = 4$ but the tails of the distribution are so heavy that the fourth moment is not finite.*

Since the distributions considered in all parts of Example 2 satisfy our Assumption A1, all graphs shown in Figure 1 are increasing in the pool size in line with Proposition 1.

3.3 VaR-Based Capital Requirements

We now turn to the VaR-based rule for the determination of the equity capital according to equation (12). The resulting default loss per policyholder is then given by

$$\bar{L}_n^v = \max(\bar{S}_n - \mathrm{VaR}_\alpha\Big[\bar{S}_n - \pi\Big] - \pi,\, 0) = \max(\bar{S}_n - Q_{1-\alpha}[\bar{S}_n],\, 0), \tag{26}$$

where we exploit equation (13) and $Q_{1-\alpha}[\bar{S}_n - \pi] + \pi = Q_{1-\alpha}[\bar{S}_n]$.

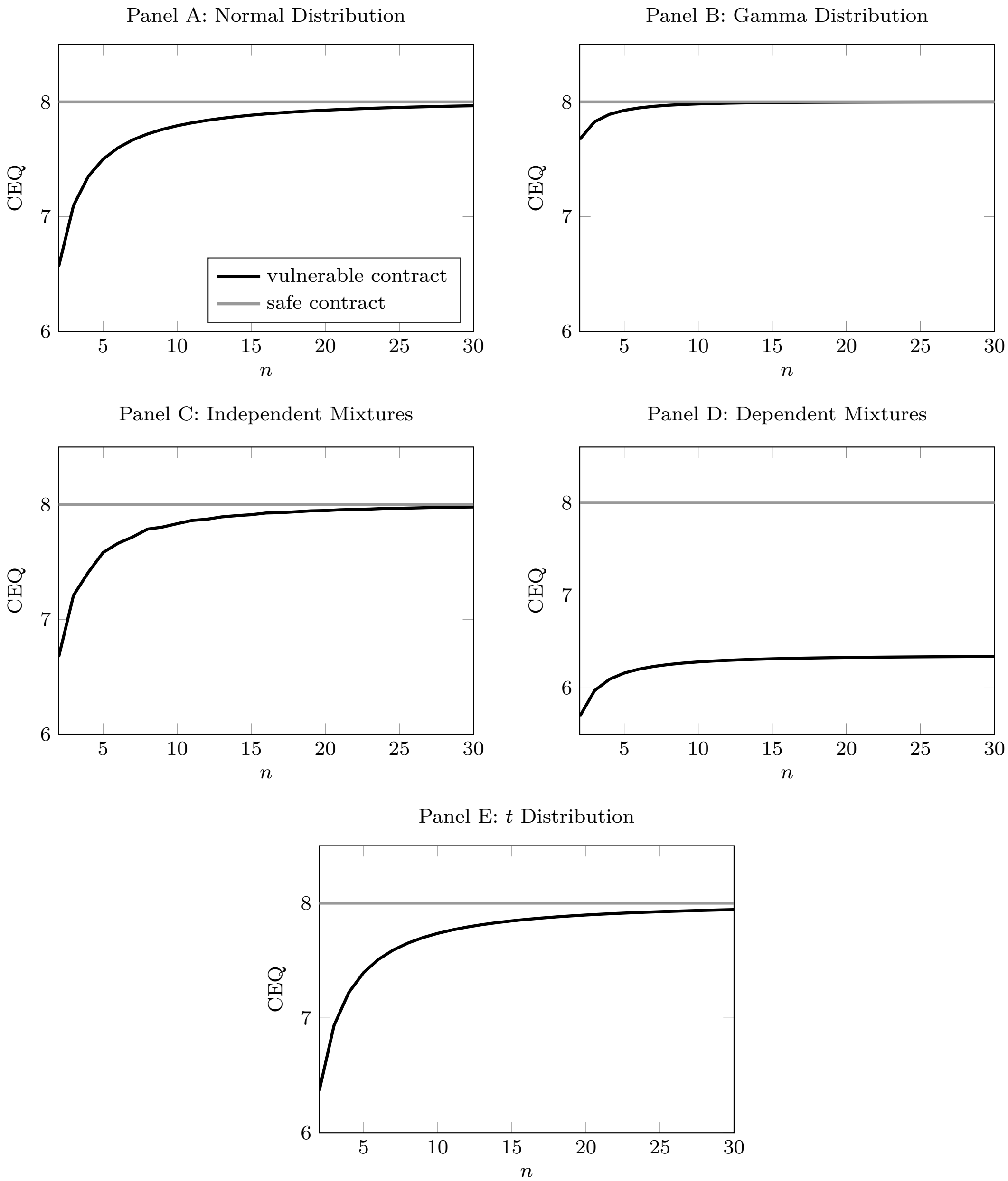
As already pointed out, this situation is highly relevant from a practical perspective given the current capital regulation in several insurance markets. Moreover, it is also of special interest from an economic point of view because a VaR-based capital rule implies in many cases that the equity per policyholder is *decreasing* in the size of the risk pool as illustrated in Example 1.

From the policyholders' perspective, a decreasing capital level can potentially offset the diversification benefits on the pool level stated in Lemma 1.[39] If this is the case, only the equity

[38] As an alternative to this specification, we present results for a truncated t distribution and power utility preferences in Section V.1 of the Online Appendix.

[39] From the potential policyholder's perspective, there is consequently a *trade-off* between diversification benefits in larger risk pools and lower levels of risk capital provided by the equity holders.

Figure 1: Volume-Based Capital Requirements



Note: This figure illustrates the relationship between the pool size and a policyholder's expected utility given volume-based capital requirements. It shows the certainty equivalent (CEQ) of buying the vulnerable contract (black line) as a function of the pool size n as well as the value of the corresponding safe contract (gray line) for the cases described in Example 2. We assume $w_{0,i} = 10$, $\pi = 2$ and $c = 1$. In Panels A-C, we present results for independent and identically distributed risks using a normal distribution with $X_i \sim \mathcal{N}(2, 4^2)$ (Panel A), a Gamma distribution with $X_i \sim \mathcal{G}(\frac{4}{3}, \frac{3}{2})$ (Panel B) and a two-component normal mixture with $\mu_L = 0$, $\mu_H = 8$, $\sigma_L = \sigma_H = 0.1$, $p_L = \frac{3}{4}$ and $p_H = \frac{1}{4}$ (Panel C). The illustration in Panel D builds on two-component normal mixtures with a common state indicator and the parameters $p_L = 0.96$, $p_H = 0.04$, $\mu_L = 5/3$, $\mu_H = 10$, $\sigma_L = 4$ and $\sigma_H = 1$. Panel E shows results for risks from a multivariate t distribution with $(X_1, \ldots, X_n)' \sim \mathcal{T}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, where $\mu_1 = \ldots = \mu_n = 2$, $(\boldsymbol{\Sigma})_{ij} = 0$, $i \neq j$, $(\boldsymbol{\Sigma})_{ii} = (4/\sqrt{2})^2$, $i = 1, \ldots, n$, and $\nu = 4$. In Panels A-D, we use an exponential utility function with $\gamma_i = 0.5$ and Panel E relies on the mean-variance preferences given in equation (25) with $\gamma_i = 1$.

holders but not the policyholders benefit from risk pooling. Interestingly, this adverse effect for policyholders is not caused by the lack of subadditivity, which the VaR is often criticized for, but it is a potential problem in cases *satisfying* the subadditivity condition. From this perspective, the given problem is related to Dhaene et al. (2008), who analyze whether risk measures can be "*too* subadditive" in the context of setting capital requirements.

The following example illustrates the different relationships between the pool size and the policyholder's utility that can arise under a VaR-based capital regulation.[40]

Example 3 *In this example, we reinvestigate the distributional assumptions and preference specifications introduced in Example 2 for the case that the available equity capital is determined using the* VaR_α *with* $\alpha = 0.05$. *Furthermore, we again choose* $\pi = 2$ *and* $w_{0,i} = 10$ *for our illustrations.*

Similar to equation (23)*, we write the expected utility from buying a vulnerable contract obtained with an exponential utility function as*

$$\mathbb{E}[u_i(W_{i,n})] = 1 - \exp\left(-\gamma_i(w_{0,i} - \pi)\right)\, \mathbb{M}_{\bar{L}_n^v}(\gamma_i), \tag{27}$$

where $\mathbb{M}_{\bar{L}_n^v}(\gamma_i)$ *is the moment generating function of* $\bar{L}_n^v$ *defined in equation* (26)*.*

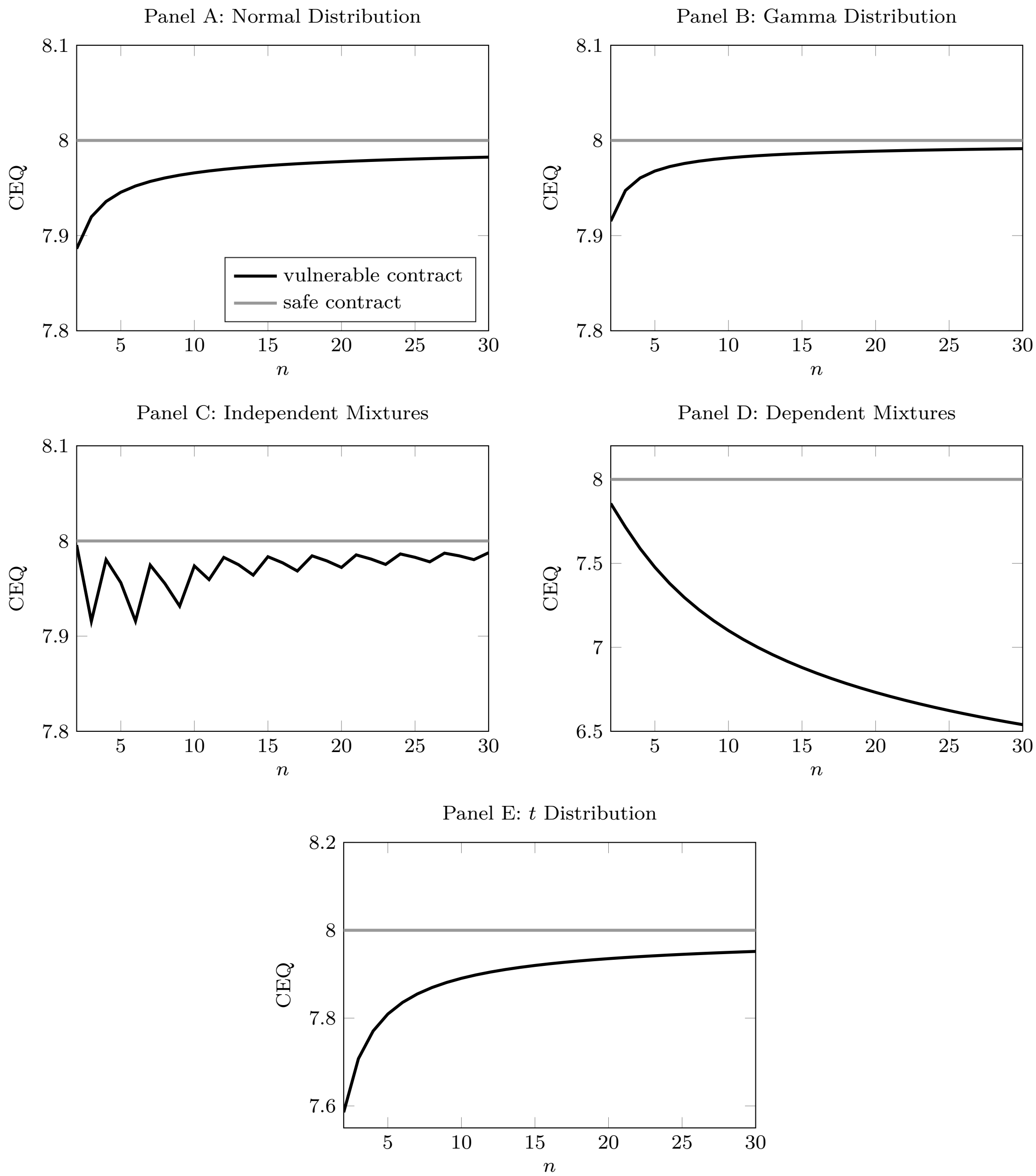
i) For the case of independent losses with a normal distribution, $X_i \sim \mathcal{N}(\mu, \sigma^2)$*, we know that* $\mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{1}{\sqrt{n}}\,\sigma\,\Phi^{-1}(1-\alpha)$ *and derive a closed form solution for* $\mathbb{M}_{\bar{L}_n^v}(\gamma_i)$ *in the Online Appendix. Building on this expression, we confirm that the policyholders' expected utility according to equation* (27) *is increasing in* n*. In Panel A of Figure 2, we graphically illustrate this increasing relationship using the same distribution parameters as in Example 2 i).*[41]

ii) For independent and identically distributed risks with a Gamma distribution, it holds that $\bar{S}_n \sim \mathcal{G}\left(n\,a, \frac{b}{n}\right)$ *and we can thus rely on the quantile function of the Gamma distribution and its partial moment generating function to calculate the expected utility from buying the vulnerable contract with an exponential utility function. We illustrate the resulting relationship between the pool size and the corresponding certainty equivalents for* $a = 4/3$ *and* $b = 3/2$ *in*

[40] Detailed calculations for this example as well as additional examples can be found in the Sections IV and V of the Online Appendix.

[41] The graph again indicates the convergence to the utility level of the safe insurance contract. See Example I.2 in the Online Appendix for a formal asymptotic analysis of the policyholders' position.

Figure 2: VaR-Based Capital Requirements



Note: This figure illustrates the relationship between the pool size and a policyholder's expected utility given VaR-based capital requirements for the distributional assumptions and preferences used in Example 3. We present results for pooling risks with a normal distribution (Panel A), a Gamma distribution (Panel B), independent mixtures (Panel C), dependent mixtures (Panel D) and a multivariate t distribution (Panel E). We show the certainty equivalent (CEQ) of buying the vulnerable contract (black line) as a function of the pool size n as well as the value of the corresponding safe contract (gray line) assuming an exponential utility function (Panels A-D) or mean-variance preferences (Panel E). Further details on our specific assumptions are provided in Figure 1.

Panel B of Figure 2. As in the case of a constant capital per policyholder, increasing the number of policies in the pool turns out to be beneficial for policyholders under these assumptions.

iii) *Next, we again consider the independent two-component mixtures used in part iii) of Example 2. We calculate* $\mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ *numerically based on the cumulative distribution function of* $\bar{S}_n$ *and use this result to determine the expected utility according to equation* (27)*. In this case, the sign of the relationship between the pool size and the policyholder's expected utility depends on the range of* n *and on the distribution parameters. For the parameters used in Example 2 iii), we illustrate the resulting relationship in Panel C of Figure 2. Although the certainty equivalent of the vulnerable contract approaches the level of the safe policy for large* n*, it can be preferable for risk-averse policyholders to participate in a smaller rather than in a larger risk pool for some values of* n*.*

iv) *We also reinvestigate the mixture specification introduced in part iv) of Example 2 that captures the occurrence of a simultaneous crash scenario for all risks in the pool with* VaR*-based capital requirements. We present the resulting relationship between the pool size and the certainty equivalent from buying the vulnerable contract in Panel D of Figure 2. This illustration shows that the expected utility from buying the vulnerable contract can even be monotonically* decreasing *in the size of the risk pool. It is interesting to note that this decrease in the expected utility occurs despite a relatively low unconditional Pearson correlation between the risks of only* 0.15*.*

v) *We finally reconsider pooling risks from a multivariate* t *distribution as in part v) of Example 2. Under this assumption, we obtain* $\mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{\sigma}{\sqrt{n}}\, q_t(1-\alpha;\nu)$ *with* $q_t(\,\cdot\,;\nu)$ *denoting the quantile function of the* t *distribution with* ν *degrees of freedom. Due to the heavy tails of the* t *distribution, we again use the mean-variance preference specification from equation* (25) *and show the resulting certainty equivalent from buying the vulnerable contract as a function of* n *in Panel E of Figure 2. Since the utility level is monotonically increasing in the pool size, the example illustrates that pooling benefits for policyholders under a* VaR*-based regulation can also occur with (moderately) heavy-tailed risks.*

The parts iii) and iv) of the previous example clearly show that risk pooling with a larger number of policies can reduce the expected utility under a VaR-based capital rule – even if Assumption A1 is satisfied. In these cases, the policyholders' position can be adversely affected by an increase in the pool size even though $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ holds for all $n \geq 1$. We thus need additional restrictions to identify cases, in which larger risk pools are beneficial for policyholders with VaR-based minimum capital requirements.

For this purpose, we next present a necessary and sufficient condition which relies on the so-called excess wealth order. According to Shaked and Shanthikumar (2007, p. 164), X is said to be smaller than Y in excess wealth order ($X \preceq_{ew} Y$) if

$$\int_{Q_u[X]}^{\infty} (1 - F_X(x))dx \leq \int_{Q_u[Y]}^{\infty} (1 - F_Y(y))dy \qquad \text{for all } u \in (0, 1). \tag{28}$$

The integrals in this definition are known as excess wealth transforms and their form can be related to the (generalized) Lorenz curve, which reveals a connection to the theory of majorization (Marshall et al., 2011, Section 17.D). In an actuarial context, excess wealth order has for example been studied by Denuit and Vermandele (1999) or Sordo (2008, 2009).

Using this stochastic order, we introduce the following assumption:

Assumption A4 $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ *for all* $n \geq 1$.

Given $\mathbb{E}\left[\bar{S}_n\right] = \mathbb{E}\left[\bar{S}_{n+1}\right]$, $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ implies $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ but the converse does not necessarily hold (Shaked and Shanthikumar, 2007, p. 166 and Theorem 4.A.34). In this sense, Assumption A4 is *more* restrictive for the average risk of the pool than Assumption A1, which only implies $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$.

To understand the economic content of Assumption A4, we characterize excess wealth order in terms of VaR and Average Value-at-Risk (AVaR). The AVaR of a random loss X at the probability level α is given by[42]

$$\mathrm{AVaR}_\alpha[X] = \frac{1}{\alpha} \int_{1-\alpha}^{1} Q_u[X]du. \tag{29}$$

[42] We follow the terminology of Föllmer and Schied (2016). The AVaR is also known as Tail Value at Risk (Dhaene et al., 2006) or Expected Shortfall (Acerbi and Tasche, 2002).

Building on this definition, we can rewrite equation (28) as follows[43]

$$\mathrm{AVaR}_\alpha[X] - \mathrm{VaR}_\alpha[X] \leq \mathrm{AVaR}_\alpha[Y] - \mathrm{VaR}_\alpha[Y] \qquad \text{for all } \alpha \in (0,1). \tag{30}$$

Assumption A4 can thus be interpreted as a decrease in the "excess tail risk" beyond the VaR on the level of the average claim per policyholder.

Using this condition, we are able to establish the following result on the benefits of risk pooling under a VaR-based capital regulation.

Proposition 2 *Suppose that the Assumptions A2 and A3 hold. Furthermore, assume that the equity capital is given by* $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$. *Then, the utility benefits of risk pooling under default risk are increasing in the pool size for all* $\alpha \in (0,1)$ *and* $n \geq 1$, *i.e.,* $W_{i,n+1} \succeq_{ssd} W_{i,n}$, *if and only if Assumption A4 is satisfied.*

Proof: See the Appendix.

According to Proposition 2, every risk-averse agent prefers pooling her risk with a larger number of policies if the excess wealth order requirement for the average claim per policyholder is satisfied.[44] Furthermore, this requirement turns out to be a necessary condition for establishing utility gains under a VaR-based regulation for all $\alpha \in (0,1)$ and $n \geq 1$.

As in the case of volume-based capital requirements, we again emphasize that the pooling benefits (or adverse pooling effects) for policyholders according to Proposition 2 exclusively focus on the diversification channel. Such risk-related utility gains (or losses) can again be offset by higher (or lower) premiums charged. Furthermore, pooling-related utility losses that can occur under the VaR-based solvency framework can alternatively be compensated by higher equity contributions. From this point of view, an important implication of Proposition 2 is that such compensations can be necessary for a larger degree of diversification and that they are related to the excess wealth order condition from Assumption A4.

To illustrate this condition, we show that it can be used to distinguish between the cases

[43] Cf. Shaked and Shanthikumar (2007, Eq. 3.C.3) or Sordo (2008, Definition 3 and Eq. 6) together with Dhaene et al. (2006, Theorem 2.1) for this characterization.

[44] An alternative sufficient condition for utility gains from risk pooling in the case of a VaR-based regulation is that $\bar{S}_{n+1}$ is smaller than $\bar{S}_n$ in dispersive order. See, e.g., Shaked and Shanthikumar (2007, p. 166) for the relationship between dispersive order and excess wealth order.

presented in Example 3.[45]

Example 4 *Under the normality assumption used in part i) of Example 3, it is straightforward to show that Assumption A4 holds. From* $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{1}{\sqrt{n}}\,\sigma\,\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]]$ *with* $Z \sim \mathcal{N}(0,1)$ *and equation* (30)*, it follows that Assumption A4 is equivalent to*

$$\frac{1}{\sqrt{n+1}}\,\sigma\left(\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] - Q_{1-\alpha}[Z]\right) \leq \frac{1}{\sqrt{n}}\,\sigma\left(\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] - Q_{1-\alpha}[Z]\right). \tag{31}$$

This inequality is satisfied for all n *and* $\alpha \in (0,1)$ *due to* $\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]] \geq Q_{1-\alpha}[Z]$*. Panel A of Figure 3 illustrates the corresponding decrease in excess tail risk for* $\alpha = 0.05$ *and the parameters used in Example 3 i). Panel B of Figure 3 also shows a decreasing relation between the excess tail risk and the pool size under the assumptions introduced in Example 3 ii), i.e., for risks with a Gamma distribution, and again* $\alpha = 0.05$*. In contrast, Panel C and Panel D of Figure 3 show that the relationship between the pool size and the excess tail risk as measured by* $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ *is* not *decreasing under the mixture assumptions used in the parts iii) and iv) of Example 3 matching the utility losses documented for these cases. For risks with a t distribution, we again find a monotonic decrease in the excess tail risk with* $\alpha = 0.05$ *as can be seen from Panel E of Figure 3. Furthermore, the same argument as in part i) applies because it holds that* $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \mu + \frac{1}{\sqrt{n}}\,\sigma\,\mathbb{E}[Z \mid Z > Q_{1-\alpha}[Z]]$ *with* $Z \sim \mathcal{T}(0,1,\nu)$*.*

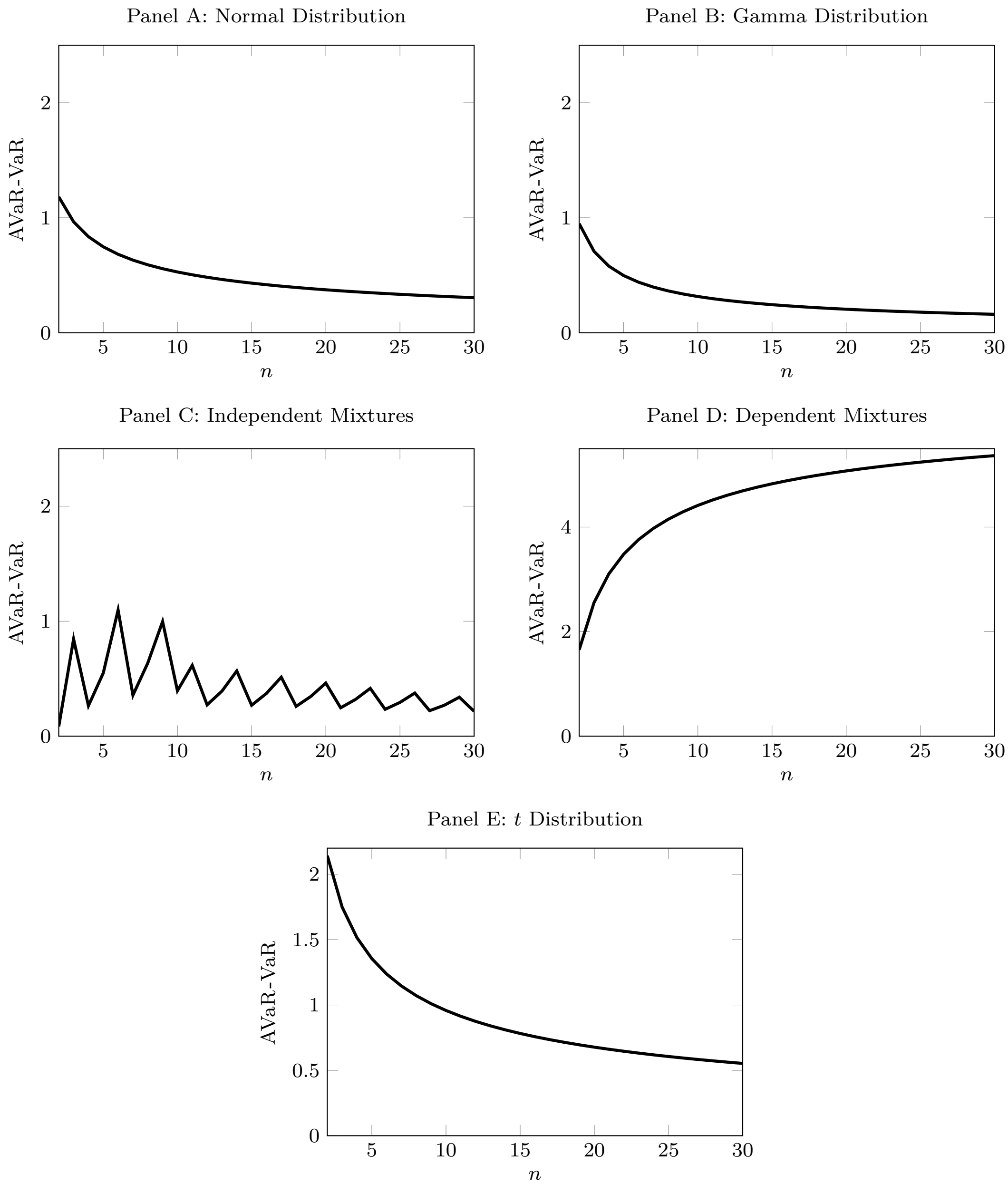
Since our excess tail risk condition is a property that depends on the marginal distributions and the stochastic dependence structure of the risks in the pool, it is not possible to replace Assumption A4 by a simpler criterion that only relies on the marginal distributions of the claims. However, motivated by the parts i) and v) of the previous example, we can relate the following more elementary assumptions on the joint distribution of the individual claims to the occurrence of pooling benefits under a VaR-based solvency regulation.

Assumption A5 $\sum_{i=1}^{n} X_i \stackrel{d}{=} a_n + b_n Z$, $a_n, b_n \in \mathbb{R}$ *with* $\mathbb{E}\left[Z^2\right] < \infty$ *for all* $n \in \mathbb{N}$*.*

Assumption A6 *The risks* $X_1, \ldots, X_n$ *are independent and identically distributed with* $X_i \stackrel{d}{=} X$ *and* X *has a stable distribution with* $\mathbb{E}[|X|] < \infty$ *(i.e., the stability index* κ *is larger than one).*

[45] We provide the relevant results for the calculation of the AVaR given our distributional assumptions in the Online Appendix.

Figure 3: VaR-Based Capital Requirements – Excess Tail Risk



Note: This figure presents the excess tail risk $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ as a function of the size of the risk pool n for $\alpha = 5\%$ and the distributional assumptions discussed in the Examples 2-4. We show results for pooling risks with a normal distribution (Panel A), a Gamma distribution (Panel B), independent mixtures (Panel C), dependent mixtures (Panel D) and a multivariate t distribution (Panel E). We refer to Figure 1 for additional details on the specific distributional assumptions for each of these cases.

Assumption A5 requires that the distribution of the total claim amount $S_n = \sum_{i=1}^{n} X_i$ is in the same location-scale family for all n and that its variance is finite.[46] Assumption A6 can be seen as an extension of Assumption A5 to risks that only satisfy $\mathbb{E}[|X_i|] < \infty$ instead of $\mathbb{E}[X^2] < \infty$ for the class of stable distributions. Using these additional assumptions, we can state the following modification of Proposition 2:

Corollary 1 *Suppose that the Assumptions A1, A2, A3 and A5 or the Assumptions A2, A3 and A6 hold. If the equity capital is given by* $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$*, then the utility benefits of risk pooling are increasing in the pool size, i.e.,* $W_{i,n+1} \succeq_{ssd} W_{i,n}$ *for all* $n \geq 1$.

Proof: See the Appendix.

Note that the conditions in Corollary 1 are only sufficient but not necessary in contrast to the excess wealth order condition (Assumption A4). An important class of distributions satisfying Assumption A5 are multivariate elliptical distributions (Owen and Rabinovitch, 1983; Landsman and Valdez, 2003; Hamada and Valdez, 2008). This class also includes the multivariate t distribution considered in our previous examples. For elliptical losses, the moment condition in Assumption A5 can be relaxed to $\mathbb{E}[|X_i|] < \infty$.[47] The applicability of Corollary 1 to heavy-tailed risks, such as risks with a multivariate t distribution[48] or risks with stable distributions, seems particularly relevant given the comprehensive empirical evidence on the occurrence of power law tails in insurance and finance.[49]

It is interesting that the results in Corollary 1 can again be related to majorization theory. If the risks are independent or uncorrelated and satisfy Assumptions A1 and A5 or Assumption A6, the preference for larger pools can be related to the majorization result from equation (18) and the Schur-convexity of the function $\psi(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i^\delta$ for $\delta > 1$ and $a_i \geq 0$, $i = 1, \ldots, n$.[50]

Finally, we note that the additional asymptotic results that we provide in the Online Appendix also apply to VaR-based capital standards (see again Proposition I.1). As for the case of a constant

[46] Note that this assumption is similar to the distributional requirements in Theorem 5 of Dhaene et al. (2008).

[47] In this case, it is straightforward to replace our variance-based reasoning with an argument using the linear aggregation properties of elliptical random vectors (McNeil et al., 2015, p. 202). We thank an anonymous referee for pointing out this extension.

[48] In Section V.1 of the Online Appendix, we present additional examples for independent risks with symmetric and asymmetric t distributions.

[49] See e.g. Embrechts et al. (1997) and the literature review in Ibragimov et al. (2015).

[50] A Schur-convex function preserves the ordering of majorization. We include a formal definition of Schur-convexity and an outline of the alternative proof in Section II of the Online Appendix. See also Ibragimov (2009), who applies a similar argument for the proof of his Theorem 4.1.

equity per policyholder, we basically find that the utility losses from default go to zero as $n \to \infty$ if the reserves are sufficiently large to cover the expected claim amount and the individual losses are independent.[51] Accordingly, the results based on standard asymptotic diversification arguments do not reflect the different consequences of pooling independent risks for the policyholders' utility documented in our analysis for finite n.

3.4 Policyholders as Owners

We eventually consider the case that the policyholders participating in a pool of size n also provide the risk capital c_n. More specifically, we assume that the policyholders and the equity holders are identical and that each of the policyholders provides the same amount of equity capital.

The total payoff that the owners obtain for providing the initial equity capital corresponds to

$$P_n = \max(c_n + \pi\, n - S_n,\, 0). \tag{32}$$

If this payoff is distributed equally, each of the owners receives

$$\bar{P}_n = \frac{1}{n} P_n = \max(\bar{c}_n + \pi - \bar{S}_n, 0) \tag{33}$$

at time $t = 1$ after paying $\bar{c}_n$ at time $t = 0$. The wealth resulting from the combined policyholder and owner position is thus given by

$$W^c_{i,n} = w_{0,i} - \pi - \bar{L}_n - \bar{c}_n + \bar{P}_n. \tag{34}$$

With $\max(a, 0) - \max(-a, 0) = a$, equation (34) simplifies to

$$W^c_{i,n} = w_{0,i} - \bar{S}_n. \tag{35}$$

This exactly corresponds to the position of the policyholders in the case of a mutual insurance company that is in detail analyzed by Gatzert and Schmeiser (2012) and Albrecht and Huggenberger (2017). Therefore, the benefits of larger risk pools documented for this case also apply

[51] Note that our asymptotic results do thus not apply to parts iv) and v) of Example 3, in which the losses being pooled are not independent.

to the given situation. In particular, we obtain from Theorem 4.1 in Albrecht and Huggenberger (2017) that larger pools are always weakly preferred to smaller pools, i.e., $W^c_{i,n+1} \succeq_{ssd} W^c_{i,n}$, if the policyholders also own the equity stake – irrespective of the premium and the available amount of equity capital. Accordingly, the potential disadvantages of risk pooling for policyholders under a VaR-based regulation are not relevant for mutual insurance companies or, more generally, if the policyholders also own an equity stake.

It is worth noting that this potential advantage of mutual insurance companies requires that the mutual insurer implements a perfect risk sharing as described by equation (35).[52] However, this position will typically not be reached in real-world insurance markets, for example due to a limited profit participation. A violation of equation (35) will typically lead to a wealth transfer between policyholder groups and/or generations.[53]

3.5 Discussion

Our results in Section 3.2 for volume-based capital requirements extend previous findings on the benefits of risk pooling for policyholders from the mutual insurance case to the case of stock insurance companies. In contrast, our results in Section 3.3 reveal that the interaction of risk pooling and VaR-based solvency rules can have adverse consequences from the policyholder's perspective. Furthermore, our analysis shows that potential adverse pooling effects are related to increases in the excess tail risk of the pool.

As already outlined following Propositions 1 and 2, these results do not necessarily imply that buying insurance from companies with portfolios of different sizes results in different levels of expected utility for policyholders. On the one hand, the owners of a company can compensate policyholders for adverse pooling effects related to default risk by lowering premiums or making larger equity contributions. On the other hand, diversification benefits for policyholders resulting from larger risk pools could be transferred to equity holders by increasing insurance premiums. All else equal, the occurrence of such adjustments in response to diversification effects likely depends on the market power that insurance companies have. In competitive markets, insurers might have to adjust premiums or equity contributions such that they offer contracts with comparable utility

[52] We thank an anonymous referee for pointing out this important limitation.

[53] Cf. Braun et al. (2015, Section 2) for a literature review on frictions relevant for comparing mutual and stock insurers and see Orozco-Garcia and Schmeiser (2019) for intergenerational wealth transfers.

levels. Moreover, an endogenous determination of the pool size could avoid or reduce adverse diversification effects for policyholders.[54]

In addition, our results in Section 3.4 show that rational policyholders could set up a mutual insurance company to avoid utility losses under a VaR-based regulation if excess tail risk increases. This alternative could force the owners of stock insurers to compensate policyholders for potential pooling-related utility losses. More generally, the alternative to set up a mutual is likely to constrain the premiums charged by stock insurers and the corresponding capital contributions such that policyholders attain similar utility levels under both organizational forms.[55]

Overall, we think it is important to pay attention to the distinct effects that diversification can have on the average risk of the pool and on the policyholder's position under VaR-based solvency rules and to be aware that potential compensation could also be required in companies with larger portfolios.

4 Extensions

We next analyze selected modifications of our baseline assumptions. In Section 4.1, we investigate the potential impact of expenses on the benefits of risk pooling in the presence of default risk. Section 4.2 introduces investment risk as an additional source of uncertainty that affects default losses. In Section 4.3, we analyze the benefits of risk pooling for more general types of coverage and more general rules for sharing the total default loss. Section 4.4 presents extensions of our results to pools with heterogeneous risks. Section 4.5 finally discusses the effect of an endogenous premium which reflects the default risk of the insurance company.

4.1 Operating Expenses

We denote the operating expenses of an insurer with a risk pool of size n by e_n. Furthermore, $\bar{e}_n := e_n/n$ is used for the average expenses per policyholder. We assume that these expenses do not grow faster than the size of the risk pool, which captures the decrease of fixed costs per

[54] A formal analysis of these effects would require additional assumptions on the structure of the insurance market and is beyond the scope of this paper.

[55] See Schmeiser and Orozco-Garcia (2021) for a comprehensive comparison of conditions (in particular, combinations of premiums and equity contributions) under which policyholders are indifferent between buying contracts offered by mutual and stock insurers.

policyholder. Accordingly, the average expenses per policyholder are weakly decreasing, which we formalize with the following assumption:

Assumption E1 $\bar{e}_n \geq \bar{e}_{n+1}$ *for all* $n \in \mathbb{N}$.

To cover the expenses, policyholders are charged the gross premium

$$\bar{\pi}_n^e = \pi + \bar{e}_n. \tag{36}$$

We thus assume that the expense loading exactly covers the expenses per policyholder. This implies that the expenses do not affect the magnitude of the default loss denoted by $\bar{L}_n^e$. Under our baseline assumptions A2 and A3, it holds that

$$\bar{L}_n^e = \max(\bar{S}_n + \bar{e}_n - \bar{c}_n - \bar{\pi}_n^e, 0) = \max(\bar{S}_n - \bar{c}_n - \pi, 0) = \bar{L}_n. \tag{37}$$

However, under Assumption E1, the gross premium is allowed to vary with the size of the risk pool. This introduces a second channel through which the size of the risk pool affects the utility of policyholders. In particular, the wealth from buying the vulnerable insurance policy accounting for the expense loading is given by

$$W_{i,n}^e = w_{0,i} - \bar{\pi}_n^e - \bar{L}_n^e = W_{i,n} - \bar{e}_n. \tag{38}$$

We can extend our results on the benefits of risk pooling to this wealth position as follows:

Corollary 2 *Suppose that the Assumptions A2, A3 and E1 hold.*

i) If the equity capital is given by $c_n = c\, n$ *and Assumption A1 is satisfied, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W_{i,n+1}^e \succeq_{ssd} W_{i,n}^e$ *for all* $n \geq 1$.

ii) If the equity capital is given by $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$ *and Assumption A4 holds, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W_{i,n+1}^e \succeq_{ssd} W_{i,n}^e$ *for all* $n \geq 1$.

Proof: See the Appendix.

Under volume-based solvency standards, we thus obtain utility gains from risk pooling without additional assumptions.[56] The cost benefits reinforce the preference for larger risk pools that comes from diversification benefits. For the case of a VaR-based capital rule, we again need the excess wealth order condition on the average claim per policyholder. In contrast to the analysis without expenses, we do not obtain an equivalence between this condition and a preference for larger risk pools. This difference reflects that the cost channel can imply a preference for larger pools even if the excess tail risk is not decreasing in the pool size.

Example 5 *To illustrate these results, we assume $e_n = f + v \cdot n$ for the total operating expenses, where $f \geq 0$ ($v \geq 0$) correspond to the fixed (variable) costs assigned to the given insurance portfolio. More specifically, we choose $f = 4$ and $v = 0.1$ for our example. We reconsider the assumptions used for Example 3 parts iii) and iv), that means, we investigate* VaR*-based capital requirements, an exponential utility function and risks that follow independent and dependent mixture distributions.*

In Figure 4, we plot the certainty equivalents of $W^e_{i,n}$ as a function of the pool size after accounting for cost benefits. Comparing Panel C of Figure 2 and Panel A of Figure 4, we find that the decrease in fixed costs per policyholder fully compensates the utility losses over the given range of n for the specification with independent risks. In contrast, we document a hump-shaped pattern in Panel B for the specification with dependent risks. In this case, the cost benefits do not overcompensate utility losses from pooling for larger n but only reduce their magnitude.

Overall, our results confirm the intuition that cost benefits of larger risk pools can be an additional channel that generates utility gains for policyholders.

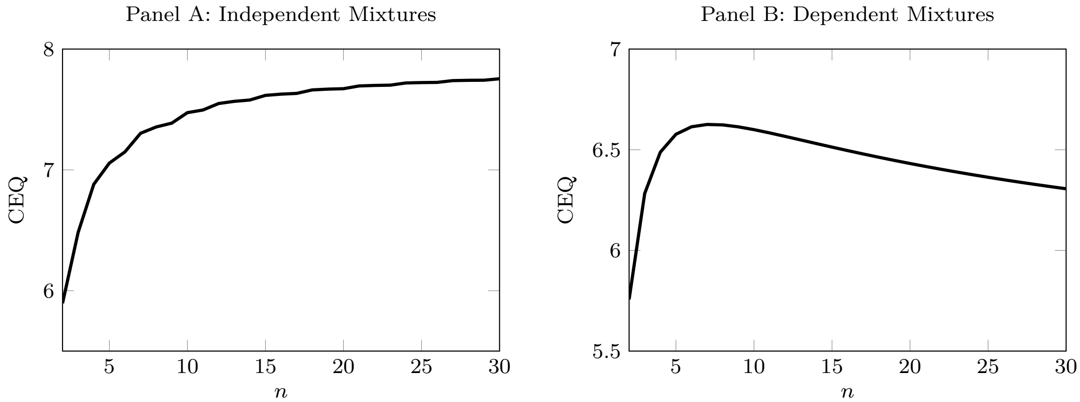
4.2 Investment Risk

We next investigate the impact of investment risk on our results. We assume that premiums are invested at a random return R and that the profits or losses from this investment reduce or increase the default loss at the end of the period. In contrast, the equity capital (minimum risk capital) may only be invested at the risk-free rate, which is still assumed to be zero.[57] The total default

[56]Note that Corollary 2 only compares the utility levels from buying a vulnerable insurance contract for different values of n. It does not include a comparison with the no insurance alternative, which can dominate an insurance solution if the operating expenses per policyholder are too high.

[57]Assuming that the equity capital c_n itself is not exposed to investment risk is a simplification, which facilitates our technical analysis.

Figure 4: Risk Pooling and Operating Expenses



Note: This figure presents the relation between the certainty equivalents of buying a vulnerable insurance contract and the pool size after accounting for decreasing operating expenses per policyholder. We rely on the assumptions introduced in Example 5. In particular, we apply VaR-based capital requirements with $\alpha = 0.05$, an exponential utility function and the mixture assumptions already used in the parts iii) and iv) of the Examples 2 and 3 for the risks in the pool. In addition, we now assume $e_n = 4{+}0.1n$ for the total operating expenses. We show the certainty equivalents (CEQ) of buying a vulnerable insurance contract as a function of the pool size n for risks with independent (Panel A) and dependent (Panel B) mixture distributions. We refer to Figure 1 for additional details on the underlying distributional assumptions used for these illustrations.

loss for a pool of size n with investment risk is then given by

$$L_n^{inv} = \max(S_n - n\,\pi\,(1+R) - c_n,\, 0) \tag{39}$$

and the average default loss corresponds to

$$\bar{L}_n^{inv} = \max(\bar{S}_n - \pi(1+R) - \bar{c}_n,\, 0). \tag{40}$$

In line with Assumption A3, we consider an equal sharing of the modified default loss among policyholders.

Assumption I1 *With investment risk, the default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = \bar{L}_n^{inv}$ for $i = 1, \ldots, n$.*

Accordingly, the default loss depends on the joint distribution of the average claim per policyholder $\bar{S}_n$ and the investment return R. We introduce the following additional assumptions on the joint distribution of R and the losses X_i, $i = 1, \ldots, n$:

Assumption I2 *The investment return R is independent of the losses $X_1, \ldots, X_n$.*

Assumption I3 *The distribution of the investment return R has a log-concave density.*

The class of distributions with log-concave densities includes important examples such as normal distributions and gamma distributions with a shape parameter $a \geq 1$, but it does not allow for heavy-tailed investment risks with power law tails.[58]

For $W_{i,n}^{inv} = w_{0,i} - \pi - \bar{L}_n^{inv}$, we are able to extend the results of our baseline analysis as follows:

Proposition 3 *Suppose that the Assumptions A2, I1 and I2 hold.*

i) If the equity capital is given by $c_n = n\,c$ and Assumption A1 is satisfied, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W_{i,n+1}^{inv} \succeq_{ssd} W_{i,n}^{inv}$ for all $n \geq 1$.

ii) If the equity capital is given by $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi(1+R)]$ and the additional Assumptions A4 and I3 hold, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W_{i,n+1}^{inv} \succeq_{ssd} W_{i,n}^{inv}$ for all $n \geq 1$.

Proof: See the Appendix.

According to part i) of this proposition, independent investment risk does not affect our result that policyholders prefer larger risk pools if the equity capital is proportional to the premiums. If the equity capital is calculated based on the VaR taking into account the additional investment risk, we do not only need the excess wealth order condition on $\bar{S}_n$ but also an additional restriction on the distribution of the investment return R to establish that larger risk pools are beneficial for policyholders. Furthermore, the conditions stated in ii) of Proposition 3 are only sufficient but not necessary in contrast to the case without investment risk considered in Proposition 2.

4.3 General Coverage and Loss Sharing Rules

We next show that our main results on the benefits of larger risk pools can also be extended to more general contract types and alternative specifications of the individual loss from default if we

[58]Cf. An (1998, Corollary 1 ii) for the tail behavior of distributions with log-concave densities and see also Bagnoli and Bergstrom (2005) for a comprehensive discussion of log-concavity and a list of distributions with log-concave densities.

replace the exchangeability assumption on the losses with the common assumption of independent and identically distributed losses. We therefore introduce:

Assumption G1 *The losses* $(X_1, \ldots, X_n)$ *are independent and identically distributed with* $\mathbb{E}[|X_i|] < \infty$ *for all* $i = 1, \ldots, n$.

In contrast to Assumption A1, G1 rules out dependent losses, which allows us to separate the impact of the own loss X_i and of the other policyholders' losses X_j, $j \neq i$, on the utility of policyholder i.

Furthermore, the full-coverage assumption can be relaxed as follows:

Assumption G2 *Agent* i *can buy an insurance contract that pays the indemnity* $f(X_i)$ *for a risk premium* $\pi > 0$, $i = 1, \ldots, n$. *The indemnity function* f *is non-decreasing and* $\mathbb{E}[|f(X_1)|] < \infty$.

In addition to full coverage, this assumption allows for well-studied contracts such as linear coverage (Mossin, 1968) or contracts with deductibles (Arrow, 1974). Note that our following analysis will take the specific form of the contract as given in contrast to the literature on insurance demand that aims at characterizing the optimal choice of f. Furthermore, Assumption G2 implies that the type of coverage, i.e. the indemnity function, is identical for all policies. In combination with Assumption G1, this ensures that we are again analyzing a homogeneous pool.

To generalize Assumption A3, we assume that the individual default loss $D_{i,n}$ depends on the own claim $f(X_i)$ and the average claim of the other policyholders

$$\bar{S}^f_{i,n} = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f(X_j). \tag{41}$$

Specifically, we introduce

$$\bar{L}^f_{i,n} = \max(\bar{S}^f_{i,n} - \pi - \bar{c}_{i,n},\, 0) \tag{42}$$

as the average default loss of the risk pool without policyholder i, where $\bar{c}_{i,n}$ is the average equity capital for the pool without policy i. This definition implicitly assumes that the pool is large and granular enough so that the claim of policyholder i is not relevant for the default of the portfolio. In other words, adding policy i to the pool does not change the occurrence of the default event. Nevertheless, the policyholder's claim can be relevant for her loss in case of a default as described by the following general default loss specification.

Assumption G3 *The default loss of policyholder i from a risk pool of size n is given by $D_{i,n} = g(\bar{L}^f_{i,n}, f(X_i))$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$, with a measurable function $g : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $g(0, y) = 0$ for $y \in \mathbb{R}$. g is (weakly) increasing in both arguments and convex in its first argument.*

A central feature of this definition is that the individual default loss is increasing in both, the own claim and the average total default loss of the other policyholders. Furthermore, the requirement $g(0, y) = 0$ for all $y \in \mathbb{R}$ ensures that the individual default loss is zero if there is no excess loss for the other policyholders, i.e., $\bar{L}^f_{i,n} = 0$. According to the convexity of g, the increase in $D_{i,n}$ is higher for high levels of $\bar{L}^f_{i,n}$. This allows for lower levels of excess losses to be partially covered by additional reserves or a guaranty fund.[59]

We briefly discuss examples for specific default functions g that satisfy Assumption G3:

Example 6 *Similar to the sharing rule used in our baseline analysis, an equal distribution of default losses corresponds to the function $g(l, y) := \beta\, l$ for $\beta \geq 0$. With this choice, we obtain*

$$D_{i,n} = \beta\, \bar{L}^f_{i,n} \tag{43}$$

for the default loss of policyholder i. A simple extension of this rule that also considers the magnitude of the own claim amount is $g(l, y) = \beta\, l \max(y, 0)$, where again $\beta \geq 0$.[60] The individual default loss is then given by

$$D_{i,n} = \beta\, \bar{L}^f_{i,n} \max(f(X_i), 0). \tag{44}$$

Accordingly, policyholders with a high claim are assigned a larger fraction of the total default loss. Note that similar rules are commonly used in insurance bankruptcy laws.[61]

We finally introduce a version of our excess wealth order condition from Assumption A4 for the given setting:

[59]The idea of introducing general "sharing rules" for the default loss is already implicit in Condition 1 of Ibragimov et al. (2010). Compared to their conditions, we do not impose an upper bound on the individual share of the default loss and we do not explicitly require that the individual default losses add up to the total excess loss. Note that the latter requirement might be less appropriate in our setting as it rules out a partial compensation of the policyholders by a guaranty fund.

[60]The max-function is not required if the loss distribution and the indemnity function imply that $f(X_i)$ is nonnegative, which is typically the case.

[61]See, e.g., the U.S. Insurer Receivership Model Act, which states that "all claims allowed within a priority class shall be paid at substantially the same percentage" (National Association of Insurance Commissioners, 2007, Section 802).

Assumption G4 $\bar{S}^f_{i,n+1} \preceq_{ew} \bar{S}^f_{i,n}$ *for all* $n \geq 1$.

Under the Assumptions G2 and G3, the wealth from buying a vulnerable insurance contract corresponds to

$$W^g_{i,n} = w_{0,i} - \pi - X_i + f(X_i) - g(\bar{L}^f_{i,n}, f(X_i)). \tag{45}$$

For this wealth position, we can establish the following results:

Proposition 4 *Suppose that the Assumptions G1, G2 and G3 hold.*

i) If the average equity capital is constant, i.e. $\bar{c}_{i,n} = c$*, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W^g_{i,n+1} \succeq_{ssd} W^g_{i,n}$ *for all* $n \geq 1$.

ii) If the average equity capital is given by $\bar{c}_{i,n} = \mathrm{VaR}_\alpha\left[\bar{S}^f_{i,n} - \pi\right]$ *and Assumption G4 holds, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e.,* $W^g_{i,n+1} \succeq_{ssd} W^g_{i,n}$ *for all* $n \geq 1$.

Proof: See the Appendix.

Proposition 4 shows that Assumption G1 is sufficient for obtaining utility gains from insurance in larger risk pools under a capital regulation with a proportional growth of the reserves. To obtain the same result under a VaR-based capital regulation, we again need an excess wealth order requirement, i.e., Assumption G4, that is analogous to Assumption A4 in our baseline analysis. Overall, Proposition 4 shows that the results of our baseline analysis obtained under a full coverage and equal default loss sharing assumption can be extended to more general contract types and to alternative rules for sharing the total default loss.

4.4 Heterogeneous Risks

We first consider the case of risks that are heterogeneous with respect to the corresponding expected loss. For this purpose, we introduce the following extension of Assumption A1:

Assumption H1 *The risks are given by* $X_i = \mu_i + Z_i$, $\mu_i \in \mathbb{R}$, $i = 1, \ldots, n$. *The distribution of* $(Z_1, \ldots, Z_n)$ *is exchangeable with* $\mathbb{E}[Z_i] = 0$.

To avoid wealth transfers between policyholders in this case, we allow for policyholder-specific premiums that can reflect differences in the expected claims.

Assumption H2 *Full coverage of X_i is offered at the individual premium π_i, $i = 1, \ldots, n$.*

Under this assumption, the premiums paid for a pool of size n are given by $\sum_{i=1}^{n} \pi_i$. Therefore, we extend our baseline assumption A3 on the default loss as follows:

Assumption H3 *The default loss of policyholder i in a pool of size n is given by $D_{i,n} = \frac{1}{n} \max\{S_n - c_n - \sum_{j=1}^{n} \pi_j, 0\}$.*

The definition of VaR-based capital requirements from equation (12) is adapted accordingly, i.e., $c_n = \mathrm{VaR}_\alpha\Big[S_n - \sum_{j=1}^{n} \pi_j\Big]$. Finally, we rewrite the excess wealth order condition in terms of $\bar{Z}_n := \frac{1}{n} \sum_{i=1}^{n} Z_i$ for the Z_i, $i = 1, \ldots, n$, given in Assumption H1.

Assumption H4 *$\bar{Z}_{n+1} \preceq_{ew} \bar{Z}_n$ for all $n \geq 1$.*

Under these assumptions, the wealth of policyholder i, who buys a vulnerable contract to fully cover X_i at the premium π_i, is given by $W_{i,n}^h = w_{0,i} - \pi_i - D_{i,n}$. For this position, we obtain the following results on the benefits of risk pooling:

Proposition 5 *Suppose that the Assumptions H1, H2, H3 hold.*

i) If the equity capital is given by $c_n = c\,n$ and the individual premiums satisfy $\pi_i = \mathbb{E}[X_i] + l$, $i = 1, \ldots, n$, then the utility benefits of risk pooling under default risk are increasing in the pool size, i.e., $W_{i,n+1}^h \succeq_{ssd} W_{i,n}^h$ for all $n \geq 1$.

ii) If the equity capital is given by $c_n = \mathrm{VaR}_\alpha\Big[S_n - \sum_{j=1}^{n} \pi_j\Big]$, then the utility benefits of risk pooling under default risk are increasing in the pool size for all $\alpha \in (0, 1)$ and all $n \geq 1$, i.e., $W_{i,n+1}^h \succeq_{ssd} W_{i,n}^h$, if and only if Assumption H4 holds.

Proof: See the Appendix.

It is interesting to note that we only need a restriction on the individual premiums for the case of proportional capital requirements. In contrast, the result for a VaR-based solvency framework does not require any additional assumption on the premiums. This reflects that changes in the total premiums paid will be offset by changes in the available equity under a VaR-based solvency rule.

Furthermore, under Assumption H1, the conditions $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ and $\bar{Z}_{n+1} \preceq_{ew} \bar{Z}_n$ are equivalent, which follows from the location independence of excess wealth order.[62]

We next consider the case of heterogeneous risks with different scale parameters.

Assumption H5 *$X_i = \mu + \sigma_i Z_i$, $\mu \in \mathbb{R}$, $\sigma_i > 0$ with $Z_i \stackrel{d}{=} Z$ and $\mathbb{E}[Z] = 0$, $i = 1, \ldots, n$.*

In this case, we investigate the following modifications of the Assumptions A5 and A6 from Section 3.3:

Assumption H6 *$\sum_{i=1}^{n} X_i \stackrel{d}{=} a_n + b_n Z$, $a_n, b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. $\mathbb{E}[Z^2] < \infty$ and $\operatorname{cov}[Z_i, Z_j] = 0$, $i \neq j$.*

Assumption H7 *$Z_1, \ldots, Z_n$ are independent and Z has a symmetric stable distribution with stability index $\kappa > 1$.*

To allow for heterogeneity in the scale parameters, we thus add the following restrictions compared to the baseline analysis: Assumption H6 requires that the risks are uncorrelated and Assumption H7 focuses on symmetric stable distributions. Under these assumptions, we can establish the following result:

Proposition 6 *Suppose that the Assumptions A2, A3 and H5 are satisfied. Furthermore, assume that the equity capital is given by $c_n = c\,n$ or $c_n = \mathrm{VaR}_\alpha[S_n - n\,\pi]$.*

i) Under Assumption H6, the utility benefits of risk pooling are increasing in the pool size, i.e., $W_{i,n+1} \succeq_{ssd} W_{i,n}$ if

$$\frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sigma_j^2 \leq \frac{1}{n^2} \sum_{j=1}^{n} \sigma_j^2. \tag{46}$$

ii) Under Assumption H7, the utility benefits of risk pooling are increasing in the pool size, i.e., $W_{i,n+1} \succeq_{ssd} W_{i,n}$ if

$$\sum_{j=1}^{n+1} \frac{\sigma_j^\kappa}{(n+1)^\kappa} \leq \sum_{j=1}^{n} \frac{\sigma_j^\kappa}{n^\kappa}. \tag{47}$$

Proof: See the Appendix.

[62] The location independence can be seen from equation (30) with $\mathrm{AVaR}_\alpha[X+a] - \mathrm{VaR}_\alpha[X+a] = \mathrm{AVaR}_\alpha[X] - \mathrm{VaR}_\alpha[X]$.

Proposition 6 presents conditions under which the pooling benefits for policyholders are robust to losses with different levels of dispersion.[63] The additional conditions in equations (46) and (47) ensure that the risk of the loss being added to the pool is not too large compared to the losses currently in the pool. Using results from majorization theory, it can be shown that these conditions are satisfied if $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq \sigma_{n+1}$ as can be seen from Ibragimov (2009, Appendix C), who studies diversification benefits for heterogeneous fat-tailed risks in a VaR-framework under this assumption.[64] Finally, it is interesting to note that the more restrictive distributional assumptions that we impose for the analysis of losses with different scale parameters are sufficient for the occurrence of pooling benefits under both the volume-based and the VaR-based solvency rule.

4.5 Actuarially Fair Default-Consistent Premium

We finally relax the assumption of a constant and exogenously given premium. In particular, we allow the premium to reflect the solvency level of the insurance company. For illustrative purposes, we consider an actuarially fair premium that takes default losses for the policyholders into account. Since the net payment to policyholder i corresponds to $X_i - D_{i,n}$, the corresponding actuarially fair premium is given by $\bar{\pi}_n^d = \mathbb{E}[X_i - D_{i,n}]$. With Assumption A3, we obtain[65]

$$\bar{\pi}_n^d = \mathbb{E}[X_i] - \mathbb{E}\Big[\max(\bar{S}_n - \bar{\pi}_n^d - \bar{c}_n, 0)\Big] , \tag{48}$$

which we refer to as default-consistent premium. We are mainly interested in analyzing whether a premium reflecting default risk can resolve the adverse pooling effects under a VaR-based regulation and, therefore, focus on $\bar{c}_n = \mathrm{VaR}_\alpha\Big[\bar{S}_n - \bar{\pi}_n^d\Big]$. Under this additional assumption, we obtain the following explicit characterization of the fair premium

$$\bar{\pi}_n^d = \mathbb{E}[X_i] - \mathbb{E}\Big[\max(\bar{S}_n - \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big] , 0)\Big] \tag{49}$$

[63] Note that Proposition 6 is again based on our baseline Assumption A2, which means that we do not have to assume policyholder-specific premiums to establish utility gains with heterogeneity in the dispersion of the risks being pooled.

[64] In particular, Ibragimov (2009) argues that the function $\chi(c_1, \ldots, c_n) = \sum_{i=1}^n \sigma_i^\kappa c_{[i]}^\kappa$ is Schur-convex for $\kappa > 1$, $c_i \geq 0$, $i = 1, \ldots, n$, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, where $c_{[i]}$ is again the i-th largest element of $(c_1, \ldots, c_n)$. We can simply apply this result for the vectors from equation (18), which satisfy $c_i = c_{[i]}$, $i = 1, \ldots, n$.

[65] This approach is similar to the fair premium analyzed by Schmeiser and Orozco-Garcia (2021).

from equation (48). Using the definition of the AVaR from equation (29), we can rewrite $\bar{\pi}_n^d$ from equation (49) as follows[66]

$$\bar{\pi}_n^d = \mathbb{E}[X_i] - \alpha \left(\mathrm{AVaR}_\alpha \left[\bar{S}_n \right] - \mathrm{VaR}_\alpha \left[\bar{S}_n \right] \right). \tag{50}$$

This representation of the actuarially fair default-consistent premium in terms of AVaR and VaR reveals an interesting relation to the discussion about excess tail risk in Section 3.3. It directly follows from the characterization of the excess wealth order in equation (30) that Assumption A4 is a sufficient condition for the default-consistent premium $\bar{\pi}_n^d$ to be increasing in the pool size. More generally, equation (50) shows that $\bar{\pi}_n^d$ resulting from a given VaR probability level α decreases in n if and only if the excess tail risk as measured by $\mathrm{AVaR}_\alpha \left[\bar{S}_n \right] - \mathrm{VaR}_\alpha \left[\bar{S}_n \right]$ increases. This decrease in the premium can at least partially compensate the policyholders for the increase in excess tail risk generated from a larger pool. In contrast, the benefits to policyholders from a decrease in excess tail risk will be partially offset by an increase in the default-consistent premium.

We illustrate this trade-off with the following example:

Example 7 *We again consider the distributional assumptions, the preference specification as well as the capital requirements introduced in Example 3. In addition, we assume that the premium for the vulnerable contract is calculated according to equation* (50)*, which implies* $W_{i,n}^d = w_{0,i} - \bar{\pi}_n^d - \bar{L}_n^v$ *for the wealth of policyholder* i.

i) We first reconsider the case of normally distributed risks. In this case, equation (50) *implies*

$$\bar{\pi}_n^d = \mu - \frac{\sigma}{\sqrt{n}} \left(\varphi(\Phi^{-1}(1-\alpha)) - \alpha\, \Phi^{-1}(1-\alpha) \right\}. \tag{51}$$

We illustrate the resulting increasing relation between n *and* $\bar{\pi}_n^d$ *in Panel A of Figure 5 using again* $X_i \sim \mathcal{N}(2, 4^2)$. *The expected utility from buying the vulnerable contract can be calculated by extending the results from Example 3 i). The corresponding certainty equivalent is shown in Panel B of Figure 5. We furthermore include the certainty equivalent of buying the vulnerable contract with a fixed premium, which we set to* $\pi = \bar{\pi}_2^d$ *to facilitate the comparison of both*

[66] See, for example, Dhaene et al. (2006, Eq. 8) with $\alpha = 1 - p$ for the relation between $\mathbb{E}[\max(Y - \mathrm{VaR}_\alpha[Y], 0)]$, $\mathrm{VaR}_\alpha[Y]$ and $\mathrm{AVaR}_\alpha[Y]$.

settings. Our results show that policyholders also benefit from increasing the size of the risk pool under a variable premium that reflects default risk but that the utility gains are somewhat lower due to the increasing price they have to pay for less risky coverage.

ii) Second, we again investigate the case of risks with a simultaneous crash scenario as modeled by the common mixture approach, for which we found a decreasing relation between the pool size and the policyholder's certainty equivalent in Example 3 iv). Consistent with the result shown in Example 4 that the excess tail risk for $\alpha = 0.05$ *is increasing in* n *under the given assumptions, we document in Panel C of Figure 5 that* $\bar{\pi}_n^d$ *is decreasing in the pool size. Accordingly, the utility losses documented for the case of a fixed premium under these assumptions in Example 3 are partially compensated by a lower price that policyholders have to pay when buying insurance from a company with a larger pool. However, as shown in Panel D of Figure 5, this effect only dampens utility losses for policyholders but does not generate a positive relation between the number of policies and the certainty equivalent.*

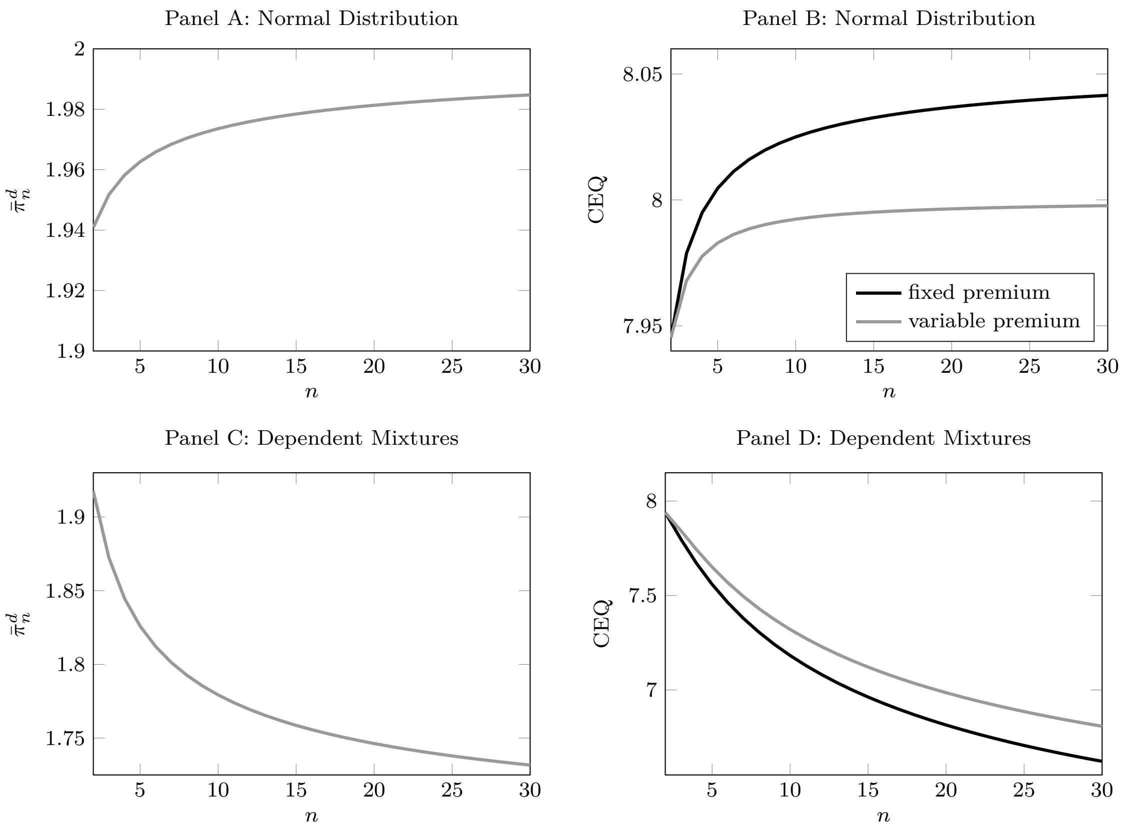
Our discussion in this section shows that a variable premium which reflects the default risk of the insurer has the potential to partly resolve the adverse pooling effects that can occur under a VaR-based regulation. However, in the examples presented above, the actuarially fair default-consistent premium only affects the magnitude of pooling effects from the policyholder's perspective without reversing their sign.

5 Concluding Remarks

We investigate the consequences of risk pooling from the policyholders' perspective under different solvency rules. We focus on the case of stock insurance companies taking into account the limited liability of equity holders. For our analysis, we assume exchangeable risks and apply an endogenous default definition. In addition, we use an SSD criterion for utility comparisons and assume that the equity capital is exogenously determined with volume-based or VaR-based capital requirements.

Under these rather general assumptions, risk pooling within a stock insurance company can affect the policyholder's position through a reduction (or an increase) of utility losses from default risk. If the equity capital grows proportionally with the pool size, all risk-averse policyholders

Figure 5: Risk Pooling with a Default-Consistent Premium



Note: This figure illustrates the effect of increasing the pool size on the fair premium reflecting the default risk of the insurance company and on the related certainty equivalent from buying the vulnerable contract under VaR-based capital requirements with $\alpha = 0.05$. We rely on the assumptions introduced in Example 7. For the illustrations in Panel A and B, we assume independent normally distributed risks. Panel A shows how the premium $\bar{\pi}_n^d$ calculated according to equation (50) varies with the size of the risk pool n and Panel B presents the certainty equivalent (CEQ) of buying a vulnerable contract sold at the default-consistent premium (gray line) $\bar{\pi}_n^d$ as a function of the pool size. Furthermore, it includes the certainty equivalent for a contract that is sold at the fixed premium (black line) $\pi = \bar{\pi}_2^d$ irrespective of the actual pool size n. In Panels C and D, we show the corresponding results for risks with a common mixture distribution. We refer to Figure 1 for additional details on the distributional assumptions and the preference specification underlying these illustrations.

benefit from risk pooling without additional assumptions. In contrast, we demonstrate that pooling a larger number of identical policies can adversely affect the policyholders' utility from insurance under a VaR-based regulation. For this case, we show that policyholders attain utility gains from larger risk pools if the excess tail risk of the average claim does not increase with the pool size.

Our analysis clarifies the effects of risk pooling on the policyholders' utility beyond the mutual insurance case analyzed by Albrecht and Huggenberger (2017). Furthermore, our findings indicate that it can be important to complement well-known asymptotic techniques with results for finite n to understand the economic implications of risk pooling and diversification. In particular, our non-asymptotic results reveal that diversification benefits on the portfolio level can have unexpected adverse consequences for policyholders under a VaR-based solvency framework. Since such pooling-related disadvantages of risk-based capital requirements can potentially be offset by lower premiums or higher equity contributions, it seems important that policyholders (and regulators) are aware of these side effects.

An interesting direction for future research can be the analysis of risk pooling from the policyholders' perspective under alternative regulatory frameworks or different assumptions that determine the available amount of equity for a given pool size.

Appendix

We first summarize a selection of results on SSD, increasing convex order and excess wealth order, which we will use in the following proofs.

Lemma 2

i) Let X and Y be two random variables. Then

$$X \preceq_{icx} Y \qquad \Leftrightarrow \qquad -X \succeq_{ssd} -Y. \tag{52}$$

ii) Let f (g) be an increasing convex (concave) function. Then, it holds for the random variables

X and Y that

$$X \preceq_{icx} Y \qquad \Rightarrow \qquad f(X) \preceq_{icx} f(Y), \tag{53}$$

$$X \preceq_{ssd} Y \qquad \Rightarrow \qquad g(X) \preceq_{ssd} g(Y). \tag{54}$$

iii) Let (X_1, X_2) and (Y_1, Y_2) be pairs of independent random variables. If $X_i \preceq_{ssd} Y_i$, $i = 1, 2$, and $w : \mathbb{R}^2 \to \mathbb{R}$ is component-wise increasing and concave, then

$$w(X_1, X_2) \preceq_{ssd} w(Y_1, Y_2). \tag{55}$$

If $X_i \preceq_{icx} Y_i$, $i = 1, 2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is component-wise increasing and convex, then

$$g(X_1, X_2) \preceq_{icx} g(Y_1, Y_2). \tag{56}$$

iv) Let X and Y be two random variables. Then

$$X \preceq_{ew} Y \quad \Leftrightarrow \quad \max(X - Q_\alpha[X], 0) \preceq_{icx} \max(Y - Q_\alpha[Y], 0) \quad \text{for all } \alpha \in (0, 1). \tag{57}$$

v) Let X and Y be random variables with $X \preceq_{ew} Y$ and let Z be a random variable that is independent of X and Y. If Z has a log-concave density, then

$$X + Z \preceq_{ew} Y + Z. \tag{58}$$

See Shaked and Shanthikumar (2007, Theorem 4.A.1) for the result in i). Part ii) and the icx-result in part iii) can be found in the Theorems 4.A.8 and 4.A.15 in Shaked and Shanthikumar (2007). The equivalence stated in equation (57) is shown by Sordo (2008, Theorem 6ii).[67] v) follows from Theorem 3.1 in Hu et al. (2006) and the relationship between excess wealth order and location independent risk order.

Proof of Proposition 1: Given Assumption A1, Lemma 1 and Lemma 2 i) imply for the

[67] A version of this result for continuous distributions can also be found in Shaked and Shanthikumar (2007, Theorem 4.A.43).

average claim amount per policyholder that $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$. Since the function $\psi(x) = \max(x - c - \pi, 0)$ is increasing and convex, it follows from Lemma 2 ii) that

$$\bar{L}^c_{n+1} = \psi(\bar{S}_{n+1}) \preceq_{icx} \psi(\bar{S}_n) = \bar{L}^c_n. \tag{59}$$

Using equation (52), we conclude $-\bar{L}^c_{n+1} \succeq_{ssd} -\bar{L}^c_n$. Note that the function ψ defined above is not strictly increasing. Therefore, we cannot establish results based on a strict version of second degree stochastic dominance with this reasoning.

For $\eta(x) := w_{0,i} - \pi + x$, it holds that $W_{i,n} = \eta(-\bar{L}^c_n)$ and $W_{i,n+1} = \eta(-\bar{L}^c_{n+1})$. Since η is an increasing linear transformation, $W_{i,n+1} \succeq_{ssd} W_{i,n}$ follows from equation (54) in Lemma 2. ■

Proof of Proposition 2: Due to the representation in equation (26), Lemma 2 iv) implies that

$$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n \qquad \Leftrightarrow \qquad \bar{L}^v_{n+1} \preceq_{icx} \bar{L}^v_n \quad \text{for all } \alpha \in (0,1). \tag{60}$$

Using Lemma 2 i), we conclude

$$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n \qquad \Leftrightarrow \qquad -\bar{L}^v_{n+1} \succeq_{ssd} -\bar{L}^v_n \quad \text{for all } \alpha \in (0,1). \tag{61}$$

Furthermore, note that $W_{i,n} = \eta(-\bar{L}^v_n)$ and $-\bar{L}^v_n = \eta^{-1}(W_{i,n})$ with $\eta(x) = w_{0,i} - \pi + x$ and $\eta^{-1}(y) = y - w_{0,i} + \pi$. Since both functions are increasing and linear (and thus concave), it follows from equations (54) and (61) that

$$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n \qquad \Leftrightarrow \qquad W_{i,n+1} \succeq_{ssd} W_{i,n} \quad \text{for all } \alpha \in (0,1). \tag{62}$$

■

Proof of Corollary 1: We have to show that the Assumptions A1 and A5 or Assumption A6 imply Assumption A4, then the result follows from Proposition 2.

We first consider the combination of Assumptions A1 and A5: Given that $\mathbb{E}[Z^2] < \infty$, we can define $Z^* := \frac{Z - \mathbb{E}[Z]}{\sigma[Z]}$. Then, it holds that $S_n \stackrel{\mathrm{d}}{=} \mathbb{E}[S_n] + \sigma[S_n] \cdot Z^*$ for all $n \in \mathbb{N}$. Due to the translation invariance and the positive homogeneity of VaR and AVaR, this implies

$$\mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \mathbb{E}\left[\bar{S}_n\right] + \sigma[\bar{S}_n] \cdot \mathrm{VaR}_\alpha[Z^*]\,, \tag{63}$$

$$\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \mathbb{E}\left[\bar{S}_n\right] + \sigma[\bar{S}_n] \cdot \mathrm{AVaR}_\alpha[Z^*]\,. \tag{64}$$

We obtain

$$\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \sigma[\bar{S}_n]\ (\mathrm{AVaR}_\alpha[Z^*] - \mathrm{VaR}_\alpha[Z^*])\,. \tag{65}$$

Therefore, Assumption A4 is satisfied due to equation (30), $\mathrm{AVaR}_\alpha[Z^*] \geq \mathrm{VaR}_\alpha[Z^*]$ and $\sigma[\bar{S}_{n+1}] \leq \sigma[\bar{S}_n]$. To see that the latter inequality holds, we note that Assumption A1 implies $\sigma[X_i] = \sigma$ for all $i = 1, \ldots, n$ and $\rho[X_i, X_j] = \rho$ for $i \neq j$. We thus have

$$\mathrm{var}\left[\bar{S}_n\right] = \mathrm{var}\left[\frac{1}{n} S_n\right] = \frac{1 + (n-1)\rho}{n}\sigma^2 \tag{66}$$

and therefore $\mathrm{var}\left[\bar{S}_{n+1}\right] \leq \mathrm{var}\left[\bar{S}_n\right]$ due to $\rho \leq 1$.

Next, suppose that Assumption A6 holds: Then, it follows from the properties of stable distributions (Nolan, 2020, Lemma 3.3) that $S_n = \sum_{i=1}^n X_i \stackrel{\mathrm{d}}{=} d_n + c_n X$ with $c_n = n^{1/\kappa}$ and thus

$$\bar{S}_n \stackrel{\mathrm{d}}{=} \frac{d_n}{n} + n^{1/\kappa - 1}\, X. \tag{67}$$

Therefore, the transformation properties of VaR and AVaR imply

$$\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right] = n^{1/\kappa - 1} \cdot (\mathrm{AVaR}_\alpha[X] - \mathrm{VaR}_\alpha[X])\,. \tag{68}$$

Since

$$\frac{dn^{1/\kappa - 1}}{dn} = (1/\kappa - 1) \cdot n^{1/\kappa - 2} < 0 \tag{69}$$

for $\kappa \in (1, 2]$ and $n > 0$, the excess tail risk is decreasing in n. ■

Proof of Corollary 2: Under Assumption E1, it holds that $-\bar{e}_{n+1} \succeq_{ssd} -\bar{e}_n$. The remaining assumptions in Corollary 2 imply $W_{i,n+1} \succeq_{ssd} W_{i,n}$ according to Proposition 1 for i) and according to Proposition 2 for ii). We can thus use equation (55) from Lemma 2 with $w(x_1, x_2) = x_1 + x_2$ to conclude

$$W^e_{i,n+1} = W_{i,n+1} - \bar{e}_{n+1} \succeq_{ssd} W_{i,n} - \bar{e}_n = W^e_{i,n}. \tag{70}$$

■

Proof of Proposition 3: We introduce $\bar{S}^{inv}_n := \bar{S}_n - \pi\, R$ for all n. Then, the average default loss with investment proceeds can be written as

$$\bar{L}^{inv}_n = \max(\bar{S}^{inv}_n - \pi - \bar{c}_n,\, 0). \tag{71}$$

For the case in i), we exploit that $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ implies

$$\bar{S}^{inv}_{n+1} \preceq_{icx} \bar{S}^{inv}_n. \tag{72}$$

This follows from the independence of $\bar{S}_n$ and $-\pi\, R$ and the closure property stated in equation (56) of Lemma 2 with $g(x_1, x_2) = x_1 + x_2$. Given equations (71) and (72), the preference for larger risk pools follows using the same arguments as in the proof of Proposition 1.

For the case in ii), we first note that log-concavity of the density of R implies that the density of $-\pi\, R$ is also log-concave (Bagnoli and Bergstrom, 2005, Corollary 5 and Theorem 8). With the Assumptions I2 and I3, we obtain

$$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n \qquad \Rightarrow \qquad \bar{S}^{inv}_{n+1} \preceq_{ew} \bar{S}^{inv}_n \tag{73}$$

from the definition of $\bar{S}^{inv}_n$ and the stability result for excess wealth order under convolutions as stated in Lemma 2 v). Moreover, it holds that

$$\bar{L}^{inv}_n = \max(\bar{S}_n - \pi\, R - Q_{1-\alpha}[\bar{S}_n - \pi\, R],\, 0) = \max(\bar{S}^{inv}_n - Q_{1-\alpha}[\bar{S}^{inv}_n],\, 0). \tag{74}$$

Using $\bar{S}^{inv}_{n+1} \preceq_{ew} \bar{S}^{inv}_n$, the SSD-ordering of the corresponding wealth positions can be derived as in the proof of Proposition 2. ■

Proof of Proposition 4: Under the Assumptions G1 and G2, $(f(X_1), \ldots, f(X_n))$ is a collection of independent and identically distributed random variables with $\mathbb{E}[|f(X_i)|] < \infty$ for all $i = 1, \ldots, n$. Therefore, we obtain from Lemma 1 that $\bar{S}^f_{i,n+1} \preceq_{icx} \bar{S}^f_{i,n}$.

We first consider the case in i), i.e. $\bar{c}_{i,n} = c$: Then, $\psi(x) = \max(x-\pi-c,\, 0)$ is non-decreasing and convex. Therefore, $\bar{L}^f_{i,n}$ defined in equation (42) satisfies $\bar{L}^f_{i,n+1} = \psi(\bar{S}^f_{i,n+1}) \preceq_{icx} \psi(\bar{S}^f_{i,n}) = \bar{L}^f_{i,n}$. With $\bar{L}^*_{i,n} = -\bar{L}^f_{i,n}$, Lemma 2 i) implies

$$\bar{L}^*_{i,n+1} = -\bar{L}^f_{i,n+1} \succeq_{ssd} -\bar{L}^f_{i,n} = \bar{L}^*_{i,n}. \tag{75}$$

Next, we introduce $h_{i,x} : \mathbb{R} \to \mathbb{R}$ with

$$h_{i,x}(l^*) = w_{0,i} - \pi - x + f(x) - g(-l^*, f(x)). \tag{76}$$

It is not difficult to see that $h_{i,x}$ is an increasing and concave function of l^* for all $x \in \mathbb{R}$. From equation (75) and Lemma 2 ii), we thus obtain $h_{i,x}(\bar{L}^*_{i,n+1}) \succeq_{ssd} h_{i,x}(\bar{L}^*_{i,n})$ for all $x \in \mathbb{R}$. Using equation (45), it follows that

$$\mathbb{E}\Big[u(W^g_{i,n+1}) \mid X_i = x\Big] = \mathbb{E}\Big[u(h_{i,x}(\bar{L}^*_{i,n+1}))\Big] \geq \mathbb{E}\Big[u(h_{i,x}(\bar{L}^*_{i,n}))\Big] = \mathbb{E}\Big[u(W^g_{i,n}) \mid X_i = x\Big] \tag{77}$$

for all increasing and concave functions u and all $x \in \mathbb{R}$. The law of iterated expectations therefore implies $W^g_{i,n+1} \succeq_{ssd} W^g_{i,n}$.[68]

Next, we turn to the case ii), i.e., $\bar{c}_{i,n} = \mathrm{VaR}_\alpha\Big[\bar{S}^f_{i,n} - \pi\Big]$: In this case, it holds that

$$\bar{L}^f_{i,n} = \max(\bar{S}^f_{i,n} - \pi - \mathrm{VaR}_\alpha\Big[\bar{S}^f_{i,n} - \pi\Big],\, 0) = \max(\bar{S}^f_{i,n} - Q_{1-\alpha}[\bar{S}^f_{i,n}],\, 0). \tag{78}$$

From Lemma 2 iv) and equation (78), we conclude

$$\bar{S}^f_{i,n+1} \preceq_{ew} \bar{S}^f_{i,n} \qquad \Leftrightarrow \qquad \bar{L}^f_{i,n+1} \preceq_{icx} \bar{L}^f_{i,n} \quad \text{for all } \alpha \in (0,1). \tag{79}$$

To prove the preference for larger risk pools, we can proceed as in part i). ■

Proof of Proposition 5: We first consider case i) with proportional capital requirements. Under the Assumptions H1 and H2, the premium can be rewritten as $\pi_i = \mathbb{E}[X_i] + l = \mu_i + l$ for all

[68] This argument is related to closure of SSD under mixtures as stated in Shaked and Shanthikumar (2007, Theorem 4.A.8).

$i = 1, \ldots, n$. With $c_n = n\,c$, we obtain

$$D_{i,n} = \frac{1}{n} \max \left\{ \sum_{j=1}^{n} (\mu_j + Z_j) - n\,c - \sum_{j=1}^{n} (\mu_j + l), 0 \right\} = \max \left\{ \bar{Z}_n - c - l, 0 \right\}. \tag{80}$$

Defining $w'_{0,i} := w_{0,i} - \mu_i$ and $\pi' := l$, we can represent the policyholder's wealth as

$$W^h_{i,n} = w_{0,i} - (\mu_i + l) - \max \left\{ \bar{Z}_n - c - l, 0 \right\} \tag{81}$$

$$= w'_{0,i} - \pi' - \max \left\{ \bar{Z}_n - c - \pi', 0 \right\}. \tag{82}$$

The result in part i) thus follows from Proposition 1 as the distribution of $(Z_1, \ldots, Z_n)$ is exchangeable.

If the available equity is given by $c_n = \mathrm{VaR}_\alpha \left[S_n - \sum_{j=1}^{n} \pi_j \right]$, then

$$D_{i,n} = \frac{1}{n} \max \left\{ S_n - \mathrm{VaR}_\alpha \left[S_n - \sum_{j=1}^{n} \pi_j \right] - \sum_{j=1}^{n} \pi_j \right\} \tag{83}$$

$$= \frac{1}{n} \max \left\{ \sum_{j=1}^{n} (\mu_j + Z_j) - \mathrm{VaR}_\alpha \left[\sum_{j=1}^{n} (\mu_j + Z_j) \right] \right\} = \max \left\{ \bar{Z}_n - \mathrm{VaR}_\alpha \left[\bar{Z}_n \right], 0 \right\}. \tag{84}$$

We can thus represent the policyholder's wealth as

$$W^h_{i,n} = w_{0,i} - \pi_i - \max \left\{ \bar{Z}_n - \mathrm{VaR}_\alpha \left[\bar{Z}_n \right], 0 \right\}. \tag{85}$$

Given this representation, the result in part ii) follows as in the proof of Proposition 2.

Proof of Proposition 6: We consider each of the Assumptions H6 and H7 individually and extend the arguments used in the proof of Corollary 1.

i) Under the Assumptions H5 and H6, we can introduce $Z^* = \frac{Z}{\sigma[Z]}$ and rely on the arguments in the equations (63)-(65). It follows that $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ if $\mathrm{var}\left[\bar{S}_{n+1}\right] \leq \mathrm{var}\left[\bar{S}_n\right]$. Since the correlations are zero, this condition is satisfied if

$$\frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sigma_j^2 \leq \frac{1}{n^2} \sum_{j=1}^{n} \sigma_j^2. \tag{86}$$

ii) Under the Assumptions H5 and H7, we obtain from results on sums of random variables with independent stable distributions (Nolan, 2020, p. 17f.) that

$$\bar{S}_n \stackrel{\mathrm{d}}{=} \mu + \left(\sum_{j=1}^{n} \frac{1}{n^\kappa} \sigma_j^\kappa\right)^{1/\kappa} Z. \tag{87}$$

Similar to equation (68), we conclude

$$\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] - \mathrm{VaR}_\alpha\left[\bar{S}_n\right] = \left(\sum_{j=1}^{n} \frac{1}{n^\kappa} \sigma_j^\kappa\right)^{1/\kappa} (\mathrm{AVaR}_\alpha[Z] - \mathrm{VaR}_\alpha[Z]). \tag{88}$$

$\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ is thus equivalent to

$$\sum_{j=1}^{n+1} \frac{1}{(n+1)^\kappa} \sigma_j^\kappa \leq \sum_{j=1}^{n} \frac{1}{n^\kappa} \sigma_j^\kappa. \tag{89}$$

We finally note that the given assumptions also ensure $\mathbb{E}\left[\bar{S}_{n+1}\right] = \mathbb{E}\left[\bar{S}_n\right]$. Therefore, $\bar{S}_{n+1} \preceq_{ew} \bar{S}_n$ implies $\bar{S}_{n+1} \preceq_{icx} \bar{S}_n$ as already noted in Section 3.3. We can thus prove the policyholder's preference for larger risk pools along the lines of the Propositions 1 and 2. ■

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Online Appendix

Risk Pooling and Solvency Regulation: A Policyholder's Perspective

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Abstract: This Online Appendix consists of five sections. In Section I, we provide an asymptotic analysis on the benefits of risk pooling for the policyholders of stock insurance companies. In Section II, we show details on the relation between our results and majorization theory. Section III presents required results on (partial) moment generating functions. In Section IV, we provide additional details on the examples included in our main analysis. Section V contains additional examples.

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I Asymptotic Results

We investigate the asymptotic impact of default risk on the policyholders' utility in a setting that is similar to our baseline analysis in Section 3.

For the specification of the contract type and the individual default loss, we again rely on the Assumptions A2 and A3. However, we focus on the standard expected utility model ($g_i(p) = p$) and we replace Assumption A1 with the following conditions on the loss distribution and the utility function.

Assumption L1 *The losses $(X_j)_{j\in\mathbb{N}}$ are identically distributed and independent with $\mathbb{E}[|X_j|] < \infty$. u_i is continuous and there is random variable $Z_i \geq 0$ with $|u_i(W_{i,n})| \leq Z_i$, for all n, and $\mathbb{E}[|Z_i|] < \infty$.*

Accordingly, we consider IID instead of exchangeable risks and additionally impose a boundedness assumption on the utility from buying the vulnerable insurance contract that avoids technical convergence problems.[1]

For our asymptotic analysis, we do not need to specify the exact dependence of the equity capital on n. Instead we only assume that the average capital per policyholder converges to a limit with $\lim_{n\to\infty} \bar{c}_n = c_a < \infty$.

Building on these assumptions, we can establish the following asymptotic result:

Proposition I.1 *Suppose that the Assumptions L1, A2 and A3 hold and that $c_a \geq \mathbb{E}[X_i] - \pi$. Then, the expected utility of the vulnerable contract converges to the utility of the safe insurance contract, i.e.,*

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = u(W_i^s). \tag{I.1}$$

Proof: We first note that a strong law of large numbers applies to $(X_i)_{i\in\mathbb{N}}$ under Assumption L1 (see, e.g., Theorem 5.17 in Klenke, 2014). Therefore, the average total claim amount converges to

[1] Alternatively, we could impose restrictions of the form $\sup\{\mathbb{E}[u(W_{i,n})], n \in \mathbb{N}\} < \infty$ and $\sup\{\mathrm{var}[u(W_{i,n})], n \in \mathbb{N}\} < \infty$ that ensure uniform integrability or we could impose a boundedness restriction on the risks that are pooled. The latter approach is common in the theory of insurance demand because it avoids bankruptcy issues on the level of the individual policyholder (Schlesinger, 2013, p. 168).

the expected claim amount, i.e.,

$$\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mathbb{E}[X_i]\,. \tag{I.2}$$

Second, the continuity of $x \mapsto \max(x - \pi - \bar{c}_n,\, 0)$ implies

$$\bar{L}_n = \max(\bar{S}_n - \pi - \bar{c}_n,\, 0) \xrightarrow{\text{a.s.}} \max(\mathbb{E}[X_i] - \pi - c_a,\, 0) = 0. \tag{I.3}$$

For the last equality, we use that $\lim_{n\to\infty} \bar{c}_n = c_a \geq \mathbb{E}[X_i] - \pi$. From equation (I.3), it follows that

$$W_{i,n} = w_{0,i} - \pi - \bar{L}_n \xrightarrow{\text{a.s.}} w_{0,i} - \pi = W_i^s. \tag{I.4}$$

Since u_i is continuous, the utility of the risky contract satisfies $u_i(W_{i,n}) \xrightarrow{\text{a.s.}} u_i(W_i^s)$.

Third, given that $|u_i(W_{i,n})|$ is bounded by an integrable random variable according to Assumption L1, we can invoke Lebesgue's convergence theorem (Klenke, 2014, Corollary 6.26) to conclude

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = \mathbb{E}\Big[\lim_{n\to\infty} u_i(W_{i,n})\Big] = u_i(W_i^s), \tag{I.5}$$

which corresponds to equation (I.1). ■

Accordingly, the disutility from the default loss asymptotically goes to zero and the expected utility of buying the vulnerable contract reaches the level of the safe insurance contract for large pools.

If the equity capital increases proportionally with the pool size, i.e., if $\bar{c}_n = c$ for all $n \in \mathbb{N}$, then $c_a \geq \mathbb{E}[X_i] - \pi$ is obviously equivalent to $c \geq \mathbb{E}[X_i] - \pi$. This result can be illustrated by reconsidering Example 2.

Example I.1 *Under the assumptions made in Example 2, the expected utility of buying a vulnerable insurance contract is given by equations* (23) *and* (IV.8)*. Furthermore, $c_a \geq \mathbb{E}[X_i] - \pi$ from Proposition I.1 implies $c \geq \mu - \pi$ and $l \geq 0$ in equation* (IV.8)*. Using these results, it is not difficult to show that $\lim_{n\to\infty} \mathbb{M}_{\bar{L}_n^c}(\gamma_i) = 1$ and thus*

$$\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = 1 - \exp(-\gamma_i\,(w_{0,i} - \pi)) = \mathbb{E}[u_i(W_i^s)]\,. \tag{I.6}$$

This convergence was illustrated in Panel A of Figure 1 in the main text.

Proposition I.1 also applies to VaR-based solvency standards. In contrast to our finite sample results, the additional excess wealth order condition is not required. To illustrate the convergence for the VaR-based case, we again consider independent normally distributed losses and an exponential utility function.

Example I.2 *Under the assumptions presented in part i) of Example 3, we can explicitly compute the limit of the expected utility from the results in equations* (27) *and* (IV.37)*. We obtain*

$$\lim_{n\to\infty} \mathbb{M}_{\bar{L}_n^v}(\gamma_i) = \exp(0) \left[1 - \Phi\left(\Phi^{-1}(1-\alpha)\right)\right] + (1-\alpha) = 1 \tag{I.7}$$

and therefore again $\lim_{n\to\infty} \mathbb{E}[u_i(W_{i,n})] = \mathbb{E}[u_i(W_i^s)]$*. This convergence was illustrated in Panel A of Figure 2 in the main text.*

In this context, it is important to note that our asymptotic results only apply to the case of independent risks. They are thus not applicable to the setting considered in part iv) of Example 3, where we documented a monotonic decrease of the policyholders' utility under a VaR-based regulation.

II Relations to Majorization Theory

In addition to the definition of the majorization ordering $\boldsymbol{a} \prec \boldsymbol{b}$ itself given in Section 3.1, we need to introduce the notion of Schur-convexity. Following Marshall et al. (2011, Definition 3.A.1), a function is Schur-convex if it preserves the ordering of majorization; that is, a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex if

$$\boldsymbol{a} \prec \boldsymbol{b} \qquad \Rightarrow \qquad \phi(\boldsymbol{a}) \leq \phi(\boldsymbol{b}). \tag{II.1}$$

In particular, every function that is symmetric and convex is also Schur-convex (Marshall et al., 2011, Proposition 3.C.2).

We first present an alternative proof of Lemma 1 based on majorization theory:

Alternative Proof of Lemma 1: We have to show that $\mathbb{E}\Big[u(-\bar{S}_{n+1})\Big] \geq \mathbb{E}\Big[u(-\bar{S}_n)\Big]$ for every increasing concave function $u : \mathbb{R} \to \mathbb{R}$ under Assumption A1. For this purpose, we rely on the result 11.B.2.C from Marshall et al. (2011), which states that the function $\psi : \mathbb{R}^n \to \mathbb{R}$ with

$$\psi(a_1, \ldots, a_n) = \mathbb{E}\left[g\left(\sum_{i=1}^{n} a_i Y_i\right)\right] \tag{II.2}$$

is symmetric and convex if the random variables $(Y_1, \ldots, Y_n)$ are exchangeable and $g : \mathbb{R} \to \mathbb{R}$ is continuous and convex. In this case, ψ is also Schur-convex. By applying this result to $g(x) = -u(x)$ and the exchangeable random variables $(-X_1, \ldots, -X_{n+1})$, we obtain

$$\boldsymbol{a} \prec \boldsymbol{b} \qquad \Rightarrow \qquad \mathbb{E}\left[-u\left(-\sum_{i=1}^{n+1} a_i X_i\right)\right] \leq \mathbb{E}\left[-u\left(-\sum_{i=1}^{n+1} b_i\, X_i\right)\right]. \tag{II.3}$$

From

$$\boldsymbol{a} = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, \frac{1}{n+1}\right) \qquad \prec \qquad \left(\frac{1}{n}, \ldots, \frac{1}{n}, 0\right) = \boldsymbol{b}, \tag{II.4}$$

we can thus conclude that

$$\mathbb{E}\left[-u\left(-\frac{1}{n+1}\sum_{i=1}^{n+1} X_i\right)\right] \leq \mathbb{E}\left[-u\left(-\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right], \tag{II.5}$$

which can be rewritten as $\mathbb{E}\Big[u(-\bar{S}_{n+1})\Big] \geq \mathbb{E}\Big[u(-\bar{S}_n)\Big]$. ■

Lemma 1 is also closely related to Proposition B.2.b in Marshall et al. (2011), which can already be found as Corollary 3 in Marshall and Proschan (1965). We next present an alternative reasoning for the proof Proposition 1, again based on majorization theory.

Alternative Proof of Proposition 1: As in the case of Lemma 1, we rely on 11.B.2.C from Marshall et al. (2011). In particular, we again exploit the Schur-convexity of the function ψ defined in equation (II.2). In this case, we choose $Y_i = X_i$, $i = 1, \ldots, n$, and define g as follows

$$g(s) := -u(w_0 - \pi - \max(s - \pi - c, 0)). \tag{II.6}$$

The convexity of this function can be seen by decomposing it into $g(s) = \phi(\eta(s))$ with

$$\phi(y) = -u(y) \qquad \text{and} \qquad \eta(s) = w_0 - \pi - \max(s - \pi - c, 0). \tag{II.7}$$

Here, (i) ϕ is decreasing and convex and (ii) η is concave (and decreasing), which implies the convexity of their composition. Under our Assumptions A2 and A3, it holds that $-\mathbb{E}[u(W_{i,n})] = \psi\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Therefore, we obtain from equation (II.4) that

$$\mathbb{E}[u(W_{i,n+1})] \geq \mathbb{E}[u(W_{i,n})] \tag{II.8}$$

for all increasing and concave utility functions u. ■

Alternative Proof of Corollary 1 for Independent or Uncorrelated Losses: Suppose that Assumptions A1 and A5 or Assumption A6 are satisfied. If the losses are independent (or uncorrelated), it can be shown that

$$\sum_{i=1}^{n} a_i X_i \stackrel{\mathrm{d}}{=} d_n + \left(\sum_{i=1}^{n} a_i^{\delta} \sigma^{\delta}\right)^{1/\delta} \cdot Z \tag{II.9}$$

with a random variable Z whose distribution does not depend on n. This follows from Nolan (2020, p. 18) for the case of stable distributions and, in this case, δ corresponds to the stability index, i.e., $\delta = \kappa$. If $\mathrm{var}[X_i] < \infty$, equation (II.9) follows from $\mathrm{var}[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i^2 \sigma^2$ with $\sigma^2 = \mathrm{var}[X_i]$ and it holds that $\delta = 2$. From the translation invariance and positive homogeneity of VaR_α and AVaR_α, we thus obtain

$$\mathrm{AVaR}_\alpha\left[\sum_{i=1}^{n} a_i X_i\right] - \mathrm{VaR}_\alpha\left[\sum_{i=1}^{n} a_i X_i\right] = \left(\sum_{i=1}^{n} a_i^{\delta}\right)^{1/\delta} \sigma\left(\mathrm{AVaR}_\alpha[Z] - \mathrm{VaR}_\alpha[Z]\right). \tag{II.10}$$

Since the function $\psi(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i^{\delta}$ is Schur-convex for $\delta > 1$ and $a_i \geq 0$, $i = 1, \ldots, n$ (Marshall et al., 2011, Proposition 3.C.1), we can use equation (II.4) to show that the excess wealth order condition in Assumption A4 is satisfied. ■

Finally, we prove the following result on the optimality of sharing the default losses equally among identical policyholders for a given pool size n.

Proposition II.1 *Suppose that Assumption A2 is satisfied. Let $L_n \geq 0$ denote the total default loss from a pool of size n and let a_i denote the fraction of L_n that is assigned to policyholder i, i.e., $D_{i,n} = a_i L_n$ with $a_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} a_i = 1$. If all policyholders have identical preferences $u_i = u$ and initial wealth levels $w_{0,i} = w_0$, $i = 1, \ldots, n$, then the loss allocation that* *maximizes the aggregate expected utility, i.e.,* $\sum_{i=1}^{n} \mathbb{E}[u(W_{i,n})]$*, is given by* $a_i = \frac{1}{n}$ *for all* $i = 1, \ldots, n$*.*

Proof of Proposition II.1: First, note that we can write the aggregate expected utility under the given assumptions as

$$\sum_{i=1}^{n} \mathbb{E}[u(W_{i,n})] = \mathbb{E}\left[\sum_{i=1}^{n} u_i(w_{0,i} - \pi - a_i\, L_n)\right] \tag{II.11}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} u(w_0 - \pi - a_i L_n)\right]. \tag{II.12}$$

Next, we introduce the function $\psi(a_1, \ldots, a_n) = -\sum_{i=1}^{n} u(w_0 - \pi - a_i l)$ with $l \in \mathbb{R}$ and compute its second derivatives:

$$\frac{\partial^2 \psi}{\partial a_i^2} = -u''(w_0 - \pi - a_i\, l)\,(-l)^2 \qquad \text{and} \qquad \frac{\partial^2 \psi}{\partial a_i \partial a_j} = 0 \quad \text{for } i \neq j. \tag{II.13}$$

From the concavity of the utility function u, it follows that $\frac{\partial^2 \psi}{\partial a_i^2} \geq 0$, such that ψ is convex for all l. By construction, ψ is also symmetric. These properties imply that the function

$$\Phi(a_1, \ldots, a_n) = \mathbb{E}[\psi(a_1, \ldots, a_n)] \tag{II.14}$$

is Schur-convex (if the expectation exists). Furthermore, according to Marshall et al. (2011, p. 9), it holds that

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (a_1, \ldots, a_n) \tag{II.15}$$

for all $a_i \geq 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} a_i = 1$. Therefore, we obtain

$$\mathbb{E}\left[\sum_{i=1}^{n} u\left(w_0 - \pi - \frac{1}{n} L_n\right)\right] \geq \mathbb{E}\left[\sum_{i=1}^{n} u(w_0 - \pi - a_i L_n)\right], \tag{II.16}$$

which concludes the proof. ■

Note that Proposition II.1 is related to the literature on optimal risk exchanges going back to the seminal work of Borch (1962). See for example Lemma 2.1 in Knispel et al. (2016) for a similar result and an overview on this literature.

III Selected Results on Moment Generating Functions

The moment generating function of a random variable X is defined as $\mathbb{M}_X(\gamma) = \mathbb{E}[\exp(\gamma\, X)]$. If X is normally distributed with expected value μ and variance σ^2, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$, it is well known that

$$\mathbb{M}_X(\gamma) = \exp\left(\gamma\,\mu + \frac{1}{2}\,\gamma^2\,\sigma^2\right). \tag{III.1}$$

If X follows a mixture of normals with K components, we can introduce a state indicator Y with $X|Y = k \sim \mathcal{N}(\mu_k, \sigma_k^2)$ and $\mathbb{P}[Y = k] = p_k$ for $k = 1, \ldots, K$. Then, the moment generating function of X can be written as

$$\begin{aligned}\mathbb{M}_X(\gamma) &= \sum_{k=1}^{K} \mathbb{E}[\exp(\gamma\, X) \mid Y = k]\ \mathbb{P}[Y = k] && \text{(III.2)}\\ &= \sum_{k=1}^{K} p_k\, \exp\left(\mu_k\,\gamma + \frac{1}{2}\gamma^2\sigma_k^2\right). && \text{(III.3)}\end{aligned}$$

We define the (upper) partial moment generating function of X as

$$\mathbb{M}_{X,t}(\gamma) = \mathbb{E}[\exp(\gamma\, X)\mathbb{1}(X > t)]\,. \tag{III.4}$$

If again $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned}\mathbb{M}_{X,t}(\gamma) &= \int_t^\infty \exp(\gamma\, x)\frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)dx && \text{(III.5)}\\ &= \int_{\tilde{t}}^\infty \exp(\gamma\,\mu)\ \exp(\gamma\,\sigma\, z)\,\varphi(z)dz && \text{(III.6)}\\ &= \int_{\tilde{t}}^\infty \exp\left(\gamma\,\mu + \frac{1}{2}\sigma^2\gamma^2\right)\ \varphi(z-\gamma\,\sigma)dz && \text{(III.7)}\\ &= \mathbb{M}_X(\gamma)\left[1 - \Phi\Big(\frac{t-\mu}{\sigma} - \gamma\cdot\sigma\Big)\right], && \text{(III.8)}\end{aligned}$$

where we use $z = \frac{x-\mu}{\sigma}$ and $\tilde{t} = \frac{t-\mu}{\sigma}$.

For the case of a normal mixture with K components, this implies

$$\mathbb{M}_{X,t}(\gamma) = \sum_{k=1}^{K} \mathbb{E}[\exp(\gamma X)\, \mathbb{1}(X > t) \mid Y = k]\ \mathbb{P}[Y = k] \tag{III.9}$$

$$= \sum_{k=1}^{K} p_k \exp\left(\mu_k \gamma + \frac{1}{2}\gamma^2\sigma_k^2\right)\left(1 - \Phi\left(\frac{t-\mu_k}{\sigma_k} - \gamma\,\sigma_k\right)\right). \tag{III.10}$$

Furthermore, we need the (upper) partial moment generating function of a random variable X with a Gamma distribution with shape parameter a and scale parameter b. In this case, we write $X \sim \mathcal{G}(a, b)$ and note that X has the probability density function

$$f_X(x) = \frac{1}{\Gamma(a)}\, b^{-a}\, x^{a-1} \exp\left(-\frac{x}{b}\right) \qquad \text{for } x \geq 0. \tag{III.11}$$

The upper partial moment generating function can thus be obtained from

$$\mathbb{M}_{X,t}(\gamma) = \int_0^\infty \exp(\gamma\, x)\, \mathbb{1}(x > t)\, \frac{1}{\Gamma(a)} b^{-a} x^{a-1} \exp\left(-\frac{x}{b}\right) dx \tag{III.12}$$

$$= \int_t^\infty \frac{1}{\Gamma(a)} b^{-a} x^{a-1} \exp\left(-\frac{x}{b}\,(1 - b\,\gamma)\right) dx. \tag{III.13}$$

For $1 - b\gamma > 0$, i.e. $\gamma < \frac{1}{b}$, we can substitute $y = x(1 - b\,\gamma)$ and obtain

$$\mathbb{M}_{X,t}(\gamma) = (1 - b\,\gamma)^{-a} \int_{t\,(1-b\,\gamma)}^\infty \frac{1}{\Gamma(a)} b^{-a} y^{a-1} \exp\left(-\frac{y}{b}\right) dy \tag{III.14}$$

$$= (1 - b\,\gamma)^{-a}\, \mathbb{P}[X > t\,(1 - b\,\gamma)] \tag{III.15}$$

$$= (1 - b\,\gamma)^{-a}\, [1 - G(t\,(1 - b\,\gamma);\, a, b)]\,, \tag{III.16}$$

where $G(\,\cdot,\, ; a, b)$ is the cdf of a Gamma distribution with parameters a and b. For $t = 0$, we obtain the well-known results $\mathbb{M}_X(\gamma) = (1 - b\,\gamma)^{-a}$.

IV Details Examples

General Remarks:

For the examples with exponential utility, we need to compute the moment generating function of $\bar{L}_n = \max(\bar{S}_n - \pi - \bar{c}_n, 0)$, which can be decomposed as follows

$$\mathbb{M}_{\bar{L}_n}(\gamma_i) = \mathbb{E}\Big[\exp\left(\gamma_i\,\bar{L}_n\right)\,\mathbb{1}\left(\bar{L}_n > 0\right)\Big] + \mathbb{E}\Big[\exp\left(\gamma_i\,\bar{L}_n\right)\right)\,\mathbb{1}\left(\bar{L}_n \le 0\right)\Big] \tag{IV.1}$$

$$= \exp(-\gamma_i\,(\pi + \bar{c}_n))\,\mathbb{E}\Big[\exp\left(\gamma_i\,\bar{S}_n\right)\mathbb{1}(\bar{S}_n > \pi + \bar{c}_n)\Big] + \mathbb{P}\Big[\bar{S}_n \le \pi + \bar{c}_n\Big] \tag{IV.2}$$

$$= \exp(-\gamma_i\,(\pi + \bar{c}_n))\,\mathbb{M}_{\bar{S}_n,\pi+\bar{c}_n}(\gamma_i) + \mathbb{P}\Big[\bar{S}_n \le \pi + \bar{c}_n\Big]\,, \tag{IV.3}$$

where $\mathbb{M}_{\bar{S}_n,\pi+\bar{c}_n}$ denotes the upper partial moment generating function of $\bar{S}_n$ for the threshold $t = \pi + \bar{c}_n$. Furthermore, we note for the case of an exponential utility function that

$$\frac{d\mathbb{E}[u_i(W_{i,n})]}{dn} = -\exp\left(-\gamma_i\,(w_{0,i} - \pi)\right)\,\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn}, \tag{IV.4}$$

which shows that $\mathbb{E}[u_i(W_{i,n})]$ is non-decreasing in n if $\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} \le 0$.

Details Example 2

i) **Normal distribution:** With $\bar{S}_n \sim \mathcal{N}(\mu, \frac{1}{n}\sigma^2)$, it holds that

$$\mathbb{P}\Big[\bar{S}_n \le \pi + \bar{c}_n\Big] = \mathbb{P}\left[Z \le \frac{\pi + \bar{c}_n - \mu}{\frac{\sigma}{\sqrt{n}}}\right] = \Phi\left(\frac{\sqrt{n}\,l_n}{\sigma}\right), \tag{IV.5}$$

where $Z \sim \mathcal{N}(0,1)$ and $l_n = \pi + \bar{c}_n - \mu$. For the computation of $\mathbb{M}_{\bar{S}_n,\pi+\bar{c}_n}(\gamma_i)$, we use equation (III.8) with

$$\frac{\pi + \bar{c}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}\,(\pi + \bar{c}_n - \mu)}{\sigma} = \frac{\sqrt{n}\,l_n}{\sigma} \qquad \text{and} \qquad -\gamma_i\,\sigma[\bar{S}_n] = -\gamma_i\,\frac{\sigma}{\sqrt{n}}. \tag{IV.6}$$

This implies

$$\mathbb{M}_{\bar{L}_n}(\gamma_i) = \exp\left(-\gamma_i\,l_n + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\cdot\left[1 - \Phi\left(\frac{\sqrt{n}\,l_n}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] + \Phi\left(\frac{\sqrt{n}\,l_n}{\sigma}\right). \tag{IV.7}$$

With $l = \pi + c - \mu$, it follows that

$$\mathbb{M}_{\bar{L}_n^c}(\gamma_i) = \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \left[1 - \Phi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] + \Phi\left(\frac{\sqrt{n}\, l}{\sigma}\right). \tag{IV.8}$$

To study the relationship between n and $\mathbb{E}[u_i(W_{i,n})]$, we compute

$$\begin{aligned}\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} &= \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2} \cdot (-1) \cdot \left[1 - \Phi^{-1}\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] \\ &\quad + \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot (-1) \cdot \varphi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right) \cdot \left(\frac{l}{2\sqrt{n}\sigma} + \frac{1}{2}\,\gamma_i\,\sigma n^{-3/2}\right) \\ &\quad + \varphi\left(\frac{\sqrt{n}\, l}{\sigma}\right) \cdot \frac{l}{2\sqrt{n}\,\sigma} \qquad \text{(IV.9)} \\ &= \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \left\{\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2} \cdot (-1) \cdot \left[1 - \Phi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] \right. \\ &\quad \left. + (-1) \cdot \varphi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right) \cdot \frac{1}{2}\,\gamma_i\,\sigma n^{-3/2}\right\} \\ &\quad + \frac{l}{2\sqrt{n}\,\sigma}\left[\varphi\left(\frac{\sqrt{n}\, l}{\sigma}\right) - \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \varphi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]. \qquad \text{(IV.10)}\end{aligned}$$

The last term vanishes because of

$$\exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \varphi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right) = \varphi\left(\frac{\sqrt{n}\, l}{\sigma}\right), \tag{IV.11}$$

which follows from the form of the density $\varphi(z)$ and

$$\begin{aligned}&-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n} - \frac{1}{2}\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)^2 \\ &\qquad = -\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n} - \frac{1}{2}\left(\frac{\sqrt{n}\, l}{\sigma}\right)^2 + \gamma_i\, l - \frac{1}{2}\gamma_i^2\,\frac{\sigma^2}{n} = -\frac{1}{2}\left(\frac{\sqrt{n}\, l}{\sigma}\right)^2. \qquad \text{(IV.12)}\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} &= \exp\left(-\gamma_i\, l + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right) \cdot \left\{\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2} \cdot (-1) \cdot \left[1 - \Phi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right] \right. \\ &\quad \left. + (-1) \cdot \varphi\left(\frac{\sqrt{n}\, l}{\sigma} - \gamma_i\,\frac{\sigma}{\sqrt{n}}\right) \cdot \frac{1}{2}\,\gamma_i\,\sigma n^{-3/2}\right\}.\end{aligned} \tag{IV.13}$$

This implies

$$\frac{d\mathbb{M}_{\bar{L}_n^c}(\gamma_i)}{dn} \leq 0 \qquad \text{and thus} \qquad \frac{d\mathbb{E}[u_i(W_{i,n})]}{dn} \geq 0 \tag{IV.14}$$

given equation (IV.4).

ii) **Gamma distribution:** If $X_i \sim \mathcal{G}(a,b)$, it follows from the well-known convolution properties of the Gamma distribution that $\bar{S}_n \sim \mathcal{G}\Big(n\,a, \frac{b}{n}\Big)$. Using equation (III.16) derived in Section III, we obtain

$$\mathbb{M}_{\bar{L}_n}(\gamma_i) = \exp(-\gamma_i\,(\pi+\bar{c}_n))\,\left(1-\frac{b}{n}\,\gamma_i\right)^{-na} \cdot \left[1 - G\bigg((\pi+\bar{c}_n)\,\left(1-\frac{b}{n}\gamma_i\right);n\,a,\frac{b}{n}\bigg)\right] + G\bigg(\pi+\bar{c}_n;n\,a,\frac{b}{n}\bigg). \tag{IV.15}$$

With $\bar{c}_n = c$, this implies

$$\mathbb{M}_{\bar{L}_n^c}(\gamma_i) = \exp(-\gamma_i\,(\pi+c))\,\left(1-\frac{b}{n}\,\gamma_i\right)^{-na}\,\left[1 - G\bigg((\pi+c)\,\left(1-\frac{b}{n}\gamma_i\right);n\,a,\frac{b}{n}\bigg)\right] + G\bigg(\pi+c;n\,a,\frac{b}{n}\bigg). \tag{IV.16}$$

iii) **Independent Mixtures:** To derive the distribution of S_n under the mixture assumption introduced in part iii), we rely on the following characterization of a normal mixture. Let $Y_1,\ldots,Y_n$ denote a series of state variables that are independent and identically distributed with $\mathbb{P}[Y_i = L] = p_L$ and $\mathbb{P}[Y_i = H] = p_H$ for all $i \in \mathbb{N}$, where $p_L + p_H = 1$. Furthermore, the distribution of X_i only depends on Y_i (X_i is independent of Y_j, $j \neq i$) and the conditional distribution of X_i is given by

$$X_i|Y_i = L \sim \mathcal{N}(\mu_L,\sigma_L^2) \qquad \text{and} \qquad X_i|Y_i = H \sim \mathcal{N}(\mu_H,\sigma_H^2). \tag{IV.17}$$

X_i is thus normally distributed conditional on Y_i with state-specific mean and variance parameters (μ_L,σ_L^2) and (μ_H,σ_H^2). Under these assumptions the unconditional distribution of X_i satisfies

$$\mathbb{P}[X_i \leq x] = p_L\,\Phi(x,\mu_L,\sigma_L^2) + p_H\,\Phi(x,\mu_H,\sigma_H^2). \tag{IV.18}$$

We next introduce an indicator for the occurrence of the high loss state

$$H_i := \mathbb{1}(Y_i = H) = \begin{cases} 0 & \text{if } Y_i = L \\ 1 & \text{if } Y_i = H \end{cases} \tag{IV.19}$$

and the counting variable $C_n = \sum_{i=1}^{n} H_i$. Then, H_i is a Bernoulli random variable with $\mathbb{P}[H_i = 1] = \mathbb{P}[Y_i = H] = p_H$ and C_n follows a Binomial distribution. This implies

$$\mathbb{P}[C_n = k] = \binom{n}{k} p_H^k (1 - p_H)^{n-k} = \binom{n}{k} p_L^{n-k} p_H^k, \qquad k = 0, \ldots, n. \tag{IV.20}$$

By construction, it holds that $\sum_{i=1}^{n} \mathbb{1}(Y_i = H) + \sum_{i=1}^{n} \mathbb{1}(Y_i = L) = n$. For the conditional distribution of S_n given that $C_n = k$, we obtain

$$S_n | C_n = k \sim \mathcal{N}((n-k)\,\mu_L + k\,\mu_H, (n-k)\,\sigma_L^2 + k\,\sigma_H^2). \tag{IV.21}$$

We conclude

$$\mathbb{P}[S_n \leq x] = \sum_{k=0}^{n} p_{n,k}\, \Phi(x, \mu_{n,k}, \sigma_{n,k}^2), \tag{IV.22}$$

where

$$\mu_{n,k} = (n-k) \cdot \mu_L + k \cdot \mu_H, \tag{IV.23}$$

$$\sigma_{n,k}^2 = (n-k) \cdot \sigma_L^2 + k \cdot \sigma_H^2, \tag{IV.24}$$

$$p_{n,k} = \binom{n}{k} \cdot p_L^{n-k} \cdot p_H^k \tag{IV.25}$$

for $k = 0, \ldots, n$. From the behavior of mixtures under linear transformations, it then follows that

$$\mathbb{P}\left[\bar{S}_n \leq x\right] = \sum_{k=0}^{n} \bar{p}_{n,k}\, \Phi(x; \bar{\mu}_{n,k}, \bar{\sigma}_{n,k}^2) \tag{IV.26}$$

with $\bar{\mu}_{n,k} = \frac{1}{n}\mu_{n,k}$, $\bar{\sigma}_{n,k}^2 = \frac{1}{n^2}\sigma_{n,k}^2$ and $\bar{p}_{n,k} = p_{n,k}$ for $k = 0, \ldots, n$. Given this distribution, we can calculate $\mathbb{M}_{\bar{L}_n^c}(\gamma_i)$ using the decomposition shown in equation (IV.3) and the general form of upper partial moment generating function for normal mixtures shown in equation (III.10).

iv) **Dependent Mixtures:** Under a mixture assumption with a common state indicator Y, the distribution of S_n is also a two-state mixture with

$$\mathbb{P}[S_n \leq x] = p_L \Phi\Big(x; n\mu_L, n\,\sigma_L^2\Big) + p_H \Phi\Big(x; n\,\mu_H; n\,\sigma_H^2\Big)\,. \tag{IV.27}$$

We can thus again evaluate the expected utility from buying a vulnerable contract based on the decomposition in equation (IV.3) and the upper partial moment generating function from equation (III.10) as in part iii) of this example.

v) t **Distribution:** To evaluate the mean-variance utility for $W_{i,n} = w_{0,i} - \pi - \max(\bar{S}_n - \pi - \bar{c}_n, 0)$, we introduce

$$\bar{S}_n^* := \bar{S}_n - \pi - \bar{c}_n \tag{IV.28}$$

and rewrite the wealth from buying a vulnerable insurance contract as

$$W_{i,n} = w_{0,i} - \pi - \mathbb{1}(\bar{S}_n^* > 0) \cdot \bar{S}_n^*. \tag{IV.29}$$

Based on this representation, we conclude that

$$\mathbb{E}[W_{i,n}] = w_{0,i} - \pi - \mathbb{E}\Big[\mathbb{1}(\bar{S}_n^* > 0) \cdot \bar{S}_n^*\Big]\,, \tag{IV.30}$$

$$\operatorname{var}[W_{i,n}] = \mathbb{E}\Big[\mathbb{1}(\bar{S}_n^* > 0) \cdot (\bar{S}_n^*)^2\Big] - \Big(\mathbb{E}\Big[\mathbb{1}(\bar{S}_n^* > 0) \cdot \bar{S}_n^*)\Big]\Big)^2\,, \tag{IV.31}$$

where the last equation follows from $\mathbb{1}(A)^2 = \mathbb{1}(A)$. We can thus compute the expected utility by calculating the first and second partial moment of $\bar{S}_n^*$.

We first show how to compute these moments for a t distribution with location parameter $m = 0$, scale parameter $s = 1$ and degree-of-freedom parameter $\nu > 2$, i.e., $Z \sim \mathcal{T}(0, 1, \nu)$. In this case, equation (3.5) from Winkler et al. (1972) and $\mathbb{E}[\mathbb{1}(Z > k)\, Z^m] = \mathbb{E}[Z^m] - \mathbb{E}[\mathbb{1}(Z \leq k)\, Z^m]$ imply

$$\mathbb{E}[\mathbb{1}(Z > k)\, Z] = \frac{(\nu + k^2)}{\nu - 1} f_t(k; \nu), \tag{IV.32}$$

$$\mathbb{E}\Big[\mathbb{1}(Z > k)\, Z^2\Big] = \frac{k\,(\nu + k^2)\, f_t(k; \nu) + \nu\,(1 - F_t(z; \nu))}{\nu - 2} \tag{IV.33}$$

with $f_t(z;\nu)$ and $F_t(z;\nu)$ denoting the probability density function and the cumulative distribution function of a t distribution with ν degrees of freedom. To account for arbitrary location and scale parameters, we note for $X = m + sZ$, $s > 0$, that

$$\mathbb{E}[\mathbb{1}(X > k)X] = m \cdot \mathbb{P}[Z > k^*] + s \cdot \mathbb{E}[\mathbb{1}(Z > k^*)Z] \tag{IV.34}$$

$$\mathbb{E}\Big[\mathbb{1}(X > k)X^2\Big] = m^2 \cdot \mathbb{P}[Z > k^*] + 2ms \cdot \mathbb{E}[\mathbb{1}(Z > k^*)Z] + s^2 \cdot \mathbb{E}\Big[\mathbb{1}(Z > k^*)Z^2\Big] \tag{IV.35}$$

with $k^* = \frac{k-m}{s}$.

In our example, it holds that $\bar{S}_n \sim \mathcal{T}(\mu, \frac{1}{n}\sigma^2, \nu)$ and thus

$$\bar{S}_n^* \sim \mathcal{T}(\mu - \pi - c, \frac{1}{n}\sigma^2, \nu). \tag{IV.36}$$

We can therefore use the results presented in equations (IV.30)-(IV.35) with $m = \mu - \pi - c$ and $s = \frac{1}{\sqrt{n}}\sigma$ to evaluate the mean-variance utility.

Details Example 3

i) **Normal Distribution:** To compute the moment generating function of $\bar{L}_n^v$, we apply equation (IV.7) with

$$l_n = \mathrm{VaR}_\alpha\Big[\bar{S}_n\Big] - \mu = \mu + \frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha) - \mu = \frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)$$

and obtain

$$\begin{aligned}\mathbb{M}_{\bar{L}_n^v}(\gamma_i) = &\exp\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha) + \frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\\ &\cdot\Big[1 - \Phi\Big(\Phi^{-1}(1-\alpha) - \gamma_i\,\frac{\sigma}{\sqrt{n}}\Big)\Big] + (1-\alpha).\end{aligned} \tag{IV.37}$$

For the derivative of this function with respect to n, it holds that

$$\begin{aligned}\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn} = &\exp\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\\ &\cdot\Bigg\{\left(\frac{1}{2}\gamma_i\sigma\,n^{-3/2}\,\Phi^{-1}(1-\alpha)-\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n^2}\right)\cdot\left[1-\Phi\left(\Phi^{-1}(1-\alpha)-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\right]\\ &-\varphi\left(\Phi^{-1}(1-\alpha)-\gamma_i\,\frac{\sigma}{\sqrt{n}}\right)\,\frac{1}{2}\,\gamma_i\,\sigma\,n^{-3/2}\Bigg\}.\end{aligned} \tag{IV.38}$$

With $\tilde{z}=\Phi^{-1}(1-\alpha)-\gamma_i\frac{\sigma}{\sqrt{n}}$, this can be rewritten as

$$\begin{aligned}\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn} = &\exp\left(-\gamma_i\,\frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha)+\frac{1}{2}\,\gamma_i^2\,\frac{\sigma^2}{n}\right)\\ &\cdot\frac{1}{2}\,\gamma_i\,\sigma\,n^{-3/2}\,\left(\tilde{z}\,(1-\Phi(\tilde{z}))-\varphi(\tilde{z})\right).\end{aligned} \tag{IV.39}$$

We next show that

$$\tilde{z}\,(1-\Phi(\tilde{z}))<\varphi(\tilde{z}). \tag{IV.40}$$

Therefore, we first note that $\varphi'(x)=-x\,\varphi(x)=-x\,\Phi'(x)$ and

$$\varphi(x)=-\int_{-\infty}^{x}s\,\Phi'(s)\,ds=-\left(\left[s\,\Phi(s)\right]_{-\infty}^{x}-\int_{-\infty}^{x}\Phi(s)\,ds\right). \tag{IV.41}$$

With $\lim_{s\to-\infty}s\,\Phi(s)=0$, this implies

$$\varphi(x)+x\,\Phi(x)=\int_{-\infty}^{x}\Phi(s)\,ds>0. \tag{IV.42}$$

From $\varphi(x)=\varphi(-x)$ and $\Phi(x)=(1-\Phi(-x))$, we obtain

$$\varphi(-x)>-x\,(1-\Phi(-x)). \tag{IV.43}$$

Choosing $x=-\tilde{z}$ proves equation (IV.40) and thus $\frac{d\mathbb{M}_{\bar{L}_n^v}(\gamma_i)}{dn}\leq 0$.

ii) **Gamma Distribution:** Denoting the quantile function of a Gamma distribution with the parameters a and b by $G^{-1}(\,\cdot\,;a,b)$, we can write $\mathrm{VaR}_\alpha\left[\bar{S}_n\right]=G^{-1}(1-\alpha\,;na,\frac{b}{n})$. For $\pi+\bar{c}_n=$

$\mathrm{VaR}_\alpha\left[\bar{S}_n\right]$, we thus obtain from equation (IV.15)

$$\mathbb{M}_{\bar{L}_n^v}(\gamma_i) = \exp\Big(-\gamma_i\, G^{-1}\Big(1-\alpha; a\, n; \frac{b}{n}\Big)\Big)\, \Big(1 - \frac{b}{n}\,\gamma_i\Big)^{-na} \Big[1 - G\Big(G^{-1}\Big(1-\alpha; a\, n; \frac{b}{n}\Big)\,\Big(1 - \frac{b}{n}\gamma_i\Big)\,; n\, a, \frac{b}{n}\Big)\Big] + (1-\alpha). \quad \text{(IV.44)}$$

iii) **Independent Mixtures:** Given the form of the distribution in equation (IV.26), the VaR_α of $\bar{S}_n$ can be determined numerically by solving

$$\sum_{k=0}^{n} \bar{p}_{n,k}\, \Phi(\mathrm{VaR}_\alpha\left[\bar{S}_n\right], \bar{\mu}_{n,k}, \bar{\sigma}^2_{n,k}) = 1 - \alpha. \quad \text{(IV.45)}$$

To calculate $\mathbb{M}_{\bar{L}_n^v}(\gamma_i)$, we again rely on equation (IV.3) in combination with equation (III.10) for $\pi + \bar{c}_n = \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$.

iv) **Dependent Mixtures:** We rely on the same arguments as in part iii) of Example 3 to compute the VaR and the moment generating function of $\bar{L}_n^v$ for the mixture distribution given in equation (IV.27). Furthermore, the unconditional covariance of the risks in the pool can be calculated as follows

$$\begin{aligned}
\operatorname{cov}[X_1, X_2] &= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\, \mathbb{E}[X_2] && \text{(IV.46)}\\
&= p_L\, \mathbb{E}[X_1\, X_2 \mid Y = L] + p_H\, \mathbb{E}[X_1\, X_2 \mid Y = H] - (\mathbb{E}[X_1])^2 && \text{(IV.47)}\\
&= p_L\, \mathbb{E}[X_1 \mid Y = L]\, \mathbb{E}[X_2 \mid Y = L] \\
&\qquad + p_H\, \mathbb{E}[X_1 \mid Y = H]\, \mathbb{E}[X_2 \mid Y = H] - \mathbb{E}[X_1]^2 && \text{(IV.48)}\\
&= p_L\, \mu_L^2 + p_H\, \mu_H^2 - (p_L\, \mu_L + p_H\, \mu_H)^2. && \text{(IV.49)}
\end{aligned}$$

v) t **Distribution:** To obtain the mean-variance utility with VaR-based capital requirements, we again use $\pi + \bar{c}_n = \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$, which implies $\bar{S}_n^* = \bar{S}_n - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]$ for $\bar{S}_n^*$ introduced in equation (IV.28) and thus

$$\bar{S}_n^* \sim \mathcal{T}(-\frac{\sigma}{\sqrt{n}}\, q_t(1-\alpha;\nu), \frac{1}{n}\sigma^2, \nu) \quad \text{(IV.50)}$$

given that $\mathrm{VaR}_\alpha\left[\bar{S}_n^*\right] = \mu + \frac{\sigma}{\sqrt{n}}\, q_t(1-\alpha;\nu)$. Accordingly, we can apply the same arguments as in part v) of Example 2 with $m = -q_t(1-\alpha;\nu)\,\frac{\sigma}{\sqrt{n}}$ and $s = \frac{\sigma}{\sqrt{n}}$ to calculate the mean-variance utility under the given assumptions.

Details Example 4

i) **Normal Distribution:** This case is fully covered in the main text.

ii) **Gamma Distribution**: To compute the $\mathrm{AVaR}_\alpha\left[\bar{S}_n\right]$, we rely again on $\bar{S}_n \sim \mathcal{G}(na, \frac{b}{n})$ and the following result from Landsman and Valdez (2005, Example 4.2). For $X \sim \mathcal{G}(a,b)$, it holds that

$$\mathrm{AVaR}_\alpha[X] = a\,b\,\frac{1 - G(\mathrm{VaR}_\alpha[X]\,;\,a+1,b)}{1 - G(\mathrm{VaR}_\alpha[X]\,;a,b)} = \frac{a\,b}{\alpha}\,\left(1 - G(\mathrm{VaR}_\alpha[X]\,;\,a+1,b)\right). \tag{IV.51}$$

Note that (i) Landsman and Valdez (2005) use the (α,β)-parametrization and (ii) $\mathrm{TCE}[X] = \mathrm{AVaR}_\alpha[X]$ for the Gamma distribution.

iii+iv) **Normal Mixtures:** Suppose that $\bar{S}_n$ follows a K-component normal mixture with the state-specific parameters μ_k and σ_k and a state indicator Y with $\mathbb{P}[Y=k] = p_k$ for $k = 1,\ldots,K$. Then, it holds that

$$\begin{aligned}\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] &= \mathbb{E}\left[\bar{S}_n \mid \bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right] &\text{(IV.52)}\\ &= \frac{1}{\alpha}\sum_{k=1}^{K} p_k\,\mathbb{E}\left[\bar{S}_n \cdot \mathbb{1}(\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]) \mid Y=k\right]. &\text{(IV.53)}\end{aligned}$$

The conditional expectations in the last expression correspond to partial expectations for normally distributed random variables. Using the results of Winkler et al. (1972, p. 292) and $\mathbb{E}[Z\,\mathbb{1}(Z \leq a)] + \mathbb{E}[Z\,\mathbb{1}(Z > a)] = \mathbb{E}[Z]$, it follows that

$$\begin{aligned}\mathbb{E}\left[\bar{S}_n \cdot \mathbb{1}(\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]) \mid Y=k\right] &= \mu_k - (-\sigma_k\,\varphi(v_k) + \mu_k\,\Phi(v_k)) &\text{(IV.54)}\\ &= \sigma_k\,\varphi(v_k) + \mu_k\,\Phi(-v_k) &\text{(IV.55)}\end{aligned}$$

with $v_k = \frac{\mathrm{VaR}_\alpha\left[\bar{S}_n\right] - \mu_k}{\sigma_k}$.

v) t **Distribution:** For $X \sim \mathcal{T}(m, s^2, \nu)$, it follows from equations (IV.32) and (IV.34) that

$$\mathrm{AVaR}_\alpha[X] = \mathbb{E}[X \mid X > Q_{1-\alpha}[X]] \tag{IV.56}$$

$$= m + s\, \frac{\nu + q_t(1-\alpha, \nu)^2}{\nu - 1}\, \frac{f_t(q_t(1-\alpha; \nu); \nu)}{\alpha}. \tag{IV.57}$$

V Additional Examples

V.1 t Distributions

In this section, we present three additional examples based on symmetric and asymmetric t distributions. These additional examples are modifications of the specification presented in part v) of the Examples 2 and 3 in the main text. In particular, we consider the following specifications:

a) We investigate the pooling effects obtained with independent risks having the same marginal distribution as the multivariate t distribution in the Examples 2 and 3. We thus assume that the risks are independent and identically distributed with $X_i \sim \mathcal{T}(\mu, \sigma^2, \nu)$, where $\mu = 2$, $\sigma = 4/\sqrt{2}$ and $\nu = 4$.

b) Our second additional example with fat-tailed risks builds on the skewed t distribution proposed by Hansen (1994). We again assume that the risks are independent and choose the following parameters $\mu = 2$, $\sigma = 4$, $\nu = 4$ and the asymmetry parameter $\lambda = 0.75$.[2] These values imply $\mathbb{E}[X_i] = 2$, $\sigma[X_i] = 4$ and a skewness coefficient of 3.63. As in our baseline example, the kurtosis is not finite.

c) We finally consider a slight variation of the assumptions used in part a), where we truncate the distribution of X_i at -20 and $+20$. Introducing these bounds on the support of X_i allows us to model the policyholder's risk preferences with a power utility function.

Furthermore, we choose $w_{0,i} = 10$, $\pi = 2$ and $c = 1$ or $\alpha = 5\%$ to determine the available equity capital as in the Examples 2 and 3. For the additional specifications in part a) and b), we again rely on the mean-variance preferences given in equation (25). In the case of the truncated t

[2] We use a different scale parameter because the density of the skewed t distribution is defined such that the scale parameter directly corresponds to the standard deviation of the distribution (for $\nu > 2$).

distribution, we apply the power utility function given by

$$u_i(w) = \frac{(w + k_i)^{1-\eta_i}}{1 - \eta_i} \tag{V.1}$$

with $k_i = 20 + \pi - w_{0,i} = 12$ and $\eta_i = 5$. In contrast to our baseline example with a multivariate t distribution, we rely on simulation techniques to determine the distribution of $\bar{S}_n$ and the corresponding certainty equivalents. In particular, we simulate 10^6 realizations of the losses $(X_1, \ldots, X_n)$ for each pool size n. Our results on the relation between the pool size and the policyholder's expected utility from buying a vulnerable contract under volume-based and VaR-based capital requirements are presented in the Figures V.1 and V.2. Similar to the corresponding baseline examples using the multivariate t distribution and mean-variance preferences, we find a positive relation between the pool size and the policyholder's utility in all three cases.

V.2 Discrete Distribution

We finally provide an additional example for the occurrence of adverse pooling effects under VaR-based capital requirements using a simple discrete loss distribution. In particular, we assume that the losses $(X_1, \ldots, X_n)$ are independent and identically distributed with

$$\mathbb{P}[X_i = 0] = p_L \qquad \text{and} \qquad \mathbb{P}[X_i = h] = p_H \tag{V.2}$$

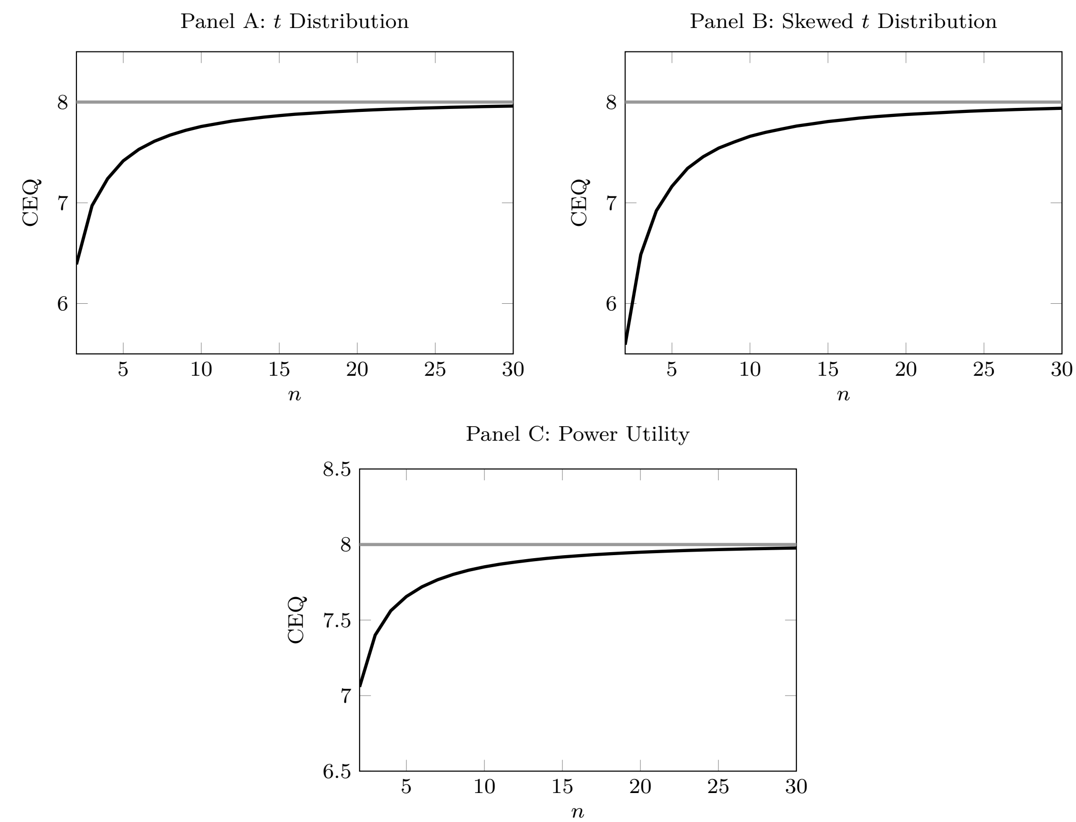
for $i = 1, \ldots, n$. Then, we can rewrite X_i as $X_i = h\, Z_i$ with a Bernoulli random variable Z_i, i.e., $\mathbb{P}[Z_i = 0] = p_L$ and $\mathbb{P}[Z_i = 1] = p_H$. Given these assumptions, $B_n := \sum_{i=1}^n Z_i$ has a Binomial distribution and it holds that

$$\bar{S}_n = \frac{k\, h}{n} \qquad \Leftrightarrow \qquad B_n = k \tag{V.3}$$

for $k = 0, \ldots, n$, which implies

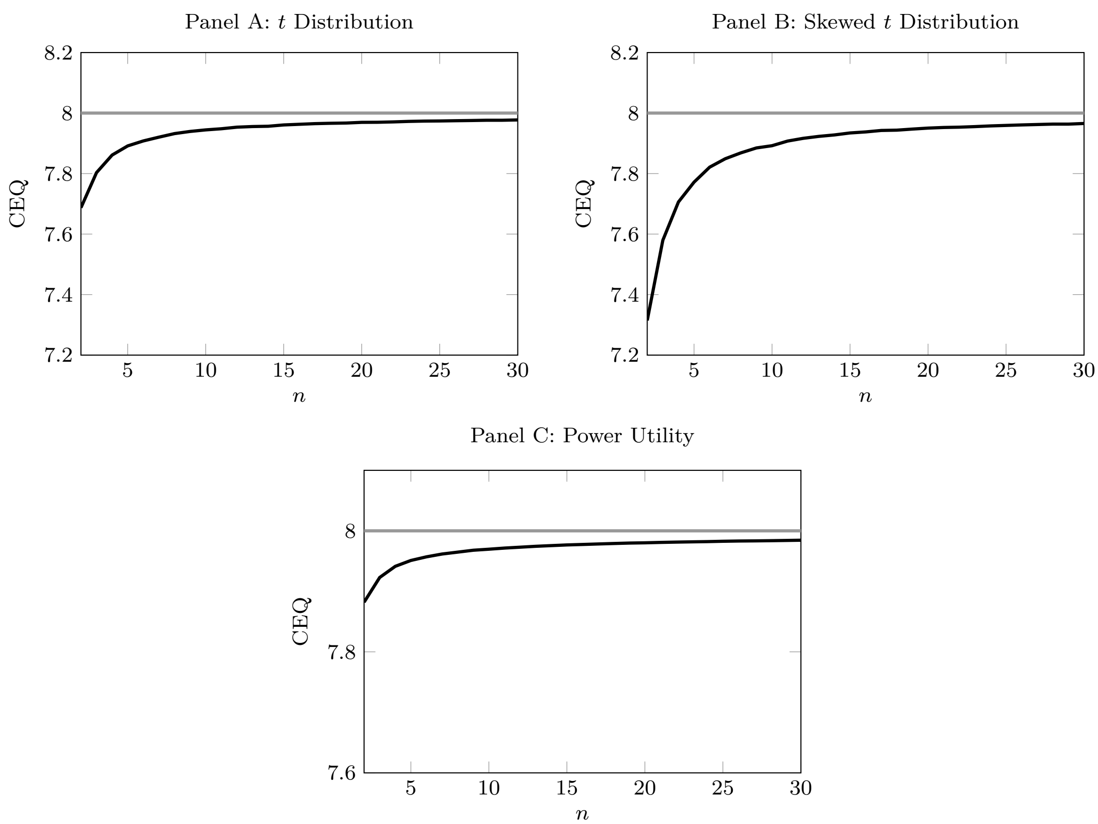
$$\mathbb{P}\left[\bar{S}_n = \frac{h\, k}{n}\right] = \mathbb{P}[B_n = k] = \binom{n}{k} \cdot p_L^{n-k} \cdot p_H^k. \tag{V.4}$$

Figure V.1: Additional Examples – Volume-Based Capital Requirements



Note: This figure complements the illustrations shown in Figure 1 on the relation between the pool size and policyholder's expected utility under volume-based capital requirements. We show the certainty equivalent (CEQ) of buying the vulnerable contract (black line) as a function of the pool size n as well as the value of the corresponding safe contract (gray line). We present three additional examples with independent and identically distributed risks. For the illustration in Panel A, we rely on a Student t distribution with $\mu = 2$, $\sigma = 4/\sqrt{2}$ and $\nu = 4$. The example shown in Panel B assumes that X_i follows the skewed t distribution proposed by Hansen (1994) with $\mu = 2$, $\sigma = 4$, $\nu = 4$ and $\lambda = 0.75$. For Panel C, we assume a t distribution with the same parameters as in Panel A but we truncate the distribution at -20 and $+20$. In the Panels A and B, we rely on mean-variance preferences according to equation (25) with $\gamma_i = 1$. The illustration in Panel C uses the certainty equivalent of a decision maker with power utility preferences according to equation (V.1) with $\eta_i = 5$.

Figure V.2: Additional Examples – VaR-Based Capital Requirements



Note: This figure complements the illustrations shown in Figure 2 on the relation between the pool size and policyholder's expected utility under VaR-based capital requirements. We show the certainty equivalent (CEQ) of buying the vulnerable contract (black line) as a function of the pool size n as well as the value of the corresponding safe contract (gray line). The additional examples rely on the same distributional assumptions and preference specifications as the corresponding illustrations in Figure V.1. We choose $\alpha = 0.05$ for the VaR_α.

Assuming an exponential utility function with the risk aversion parameter γ_i, we obtain

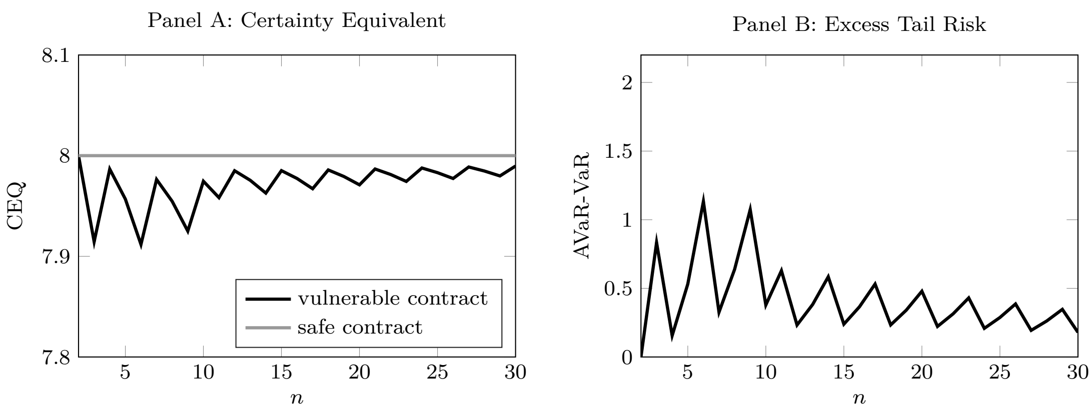
$$\mathbb{E}[u_i(W_{i,n})] = 1 - \sum_{k=0}^{n} \mathbb{P}\left[\bar{S}_n = \frac{h\,k}{n}\right] \exp\left(-\gamma_i \left(w_{0,i} - \pi - \max\left\{\frac{h\,k}{n} - \mathrm{VaR}_\alpha\left[\bar{S}_n\right]; 0\right\}\right)\right) \tag{V.5}$$

for the expected utility from a buying a vulnerable insurance contract from a company with a total risk pool of size n. To determine the VaR_α of $\bar{S}_n$, we simply use the general VaR definition for the probability distribution given in equation (V.4). The corresponding AVaR_α can be computed as follows

$$\begin{aligned}\mathrm{AVaR}_\alpha\left[\bar{S}_n\right] = \frac{1}{\alpha}\Big\{&\mathbb{E}\left[\bar{S}_n \mathbb{1}\left(\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right)\right] \\ &+ \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\left(\alpha - \mathbb{P}\left[\bar{S}_n > \mathrm{VaR}_\alpha\left[\bar{S}_n\right]\right]\right)\Big\}.\end{aligned} \tag{V.6}$$

In this case, the relationship between the pool size and the expected utility depends on the distribution parameters. We illustrate the expected utility and the excess tail risk as a function of n for $h = 8$, $p_L = \frac{3}{4}$ (and thus $p_H = \frac{1}{4}$), $\pi = 2$, $w_{0,i} = 10$, $\gamma_i = 0.5$ and $\alpha = 5\%$ in Figure V.3. With these parameters, the results are qualitatively similar to the setting with independent mixtures of normals that we discussed as Example 3 iii). Note, however, that the change in the policyholders' utility level is partly associated with a varying default probability, which is caused by the discreteness of the loss distribution.

Figure V.3: Additional Example – Binomial Distribution



Note: This figure illustrates the case of VaR-based capital requirements under the assumption of independent and identically distributed losses with $\mathbb{P}[X_i = 0] = 0.75$ and $\mathbb{P}[X_i = 8] = 0.25$. Furthermore, we assume an exponential utility function with $\gamma_i = 0.5$ and suppose that $w_{0,i} = 10$, $\pi = 2$ and $\alpha = 0.05$. Panel A depicts the certainty equivalents (CEQ) of buying the safe insurance contract (gray line) and the vulnerable insurance contract (black line) as a function of the pool size n. Panel B presents the excess tail risk $\text{AVaR}_\alpha\left[\bar{S}_n\right] - \text{VaR}_\alpha\left[\bar{S}_n\right]$ as a function of n.

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